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Volume XV - Application of Optimization Techniques

by A. S. Abbott and A. L. Blackford

Prepared by
NORTH AMERICAN AVIATION, INC.
Downey, Calif.
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FOREWORD

This report was prepared under contract NAS 8-11495 and is one of a series intended to illustrate analytical methods used in the fields of Guidance, Flight Mechanics, and Trajectory Optimization. Derivations, mechanizations and recommended procedures are given. Below is a complete list of the reports in the series.

Volume I	Coordinate Systems and Time Measure
Volume II	Observation Theory and Sensors
Volume III	The Two Body Problem
Volume IV	The Calculus of Variations and Modern Applications
Volume V	State Determination and/or Estimation
Volume VI	The N-Body Problem and Special Perturbation Techniques
Volume VII	The Pontryagin Maximum Principle
Volume VIII	Boost Guidance Equations
Volume IX	General Perturbations Theory
Volume X	Dynamic Programming
Volume XI	Guidance Equations for Orbital Operations
Volume XII	Relative Motion, Guidance Equations for Terminal Rendezvous
Volume XIII	Numerical Optimization Methods
Volume XIV	Entry Guidance Equations
Volume XV	Application of Optimization Techniques
Volume XVI	Mission Constraints and Trajectory Interfaces
Volume XVII	Guidance System Performance Analysis

The work was conducted under the direction of C. D. Baker, J. W. Winch, and D. P. Chandler, Aero-Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The North American program was conducted under the direction of H. A. McCarty and G. E. Townsend.

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1.0 STATEMENT OF THE PROBLEM

Several of the earlier monographs in this series dealt with the theoretical and numerical aspects of optimization theory. [See References 1.3, 1.4, 1.5, and 1.9.] The Pontryagin Maximum Principle, Dynamic Programming, and the Calculus of Variations have been presented in connection with both deterministic and stochastic problems, and the numerical procedures which are available for constructing computer solutions have been treated. This Monograph is intended to consolidate these previous results and show applications of the methods of optimization theory by illustrating their mechanizations for a variety of problems. In keeping with the general tone of the monograph series, the applications considered are of the trajectory and guidance type encountered in the aerospace industry.

While trajectory and guidance applications constitute but one small area of interest in optimization theory, the area is still too large to be treated in a document such as this with any degree of completeness. Consequently, the material which follows is the result of a selection process designed to provide more of a "survey" than an "in-depth" treatment. Problems were selected from those available in the literature with the emphasis on those which utilized basic dynamic equations so that the optimization formulation is not obscured by the equations of motion. An attempt was made for most of the problems considered to include a formulation simple enough to allow an analytic solution along with a more general formulation which requires a numerical technique to obtain a solution. This dual formulation was made to acquaint the reader with the general nature of the simplifying assumptions necessary to obtain an analytic solution. The problems have been rather loosely grouped into those relating to boost or boost type problems [Section (2.2)] and those relating to orbital maneuvers [Section (2.3)]. Although it is pointed out in Section (2.1) that both the Calculus of Variations and the Pontryagin Maximum Principle lead to the same set of equations, the application of both techniques is illustrated in Section (2.2) with the choice of the particular technique found in the literature. In Section (2.3), however, all the optimization has been couched in the language of the Pontryagin Maximum Principle regardless of the technique used in the literature. This approach was taken because the problems of Section (2.2) are more closely related than those of Section (2.2); and, while it is felt that both methods should be illustrated [as in Section (2.2)], a certain amount of continuity is maintained in Section (2.3) by the use of only one technique. In both Sections (2.2) and (2.3) rather general, non-linear problems have been formulated and the solution process discussed. If there are approximations which will allow an analytic solution, they are introduced; if not (or if a solution to the general problem is desired), a particular numerical technique which can be used to obtain a solution is discussed.

In general, the problems discussed are required to extremize some function of the terminal state such as time, final mass, altitude, etc. Some specific problems considered in Section (2.2) are the maximization of altitude for a rocket shot vertically, the optimization of a pay-off function which contains combinations of the components of the final state for a two-dimensional rocket flight, minimization of the fuel required to achieve a soft landing, and the determination of the staging points for a multi-stage rocket. In these problems, the optimization is performed by varying the direction and magnitude of the thrust vector or, in the last case mentioned, the time at which staging occurs. In the section on orbital problems, attention is directed to making transfers from one orbit to another, while minimizing final time or fuel, and to accomplishing a rendezvous between two space vehicles while minimizing the same sort of functions of the terminal state. For these problems, the optimization is accomplished by correct selection of a thrust magnitude-direction program.

2.0 STATE-OF-THE-ART

2.1 The Optimization Problem and Its Formulation

Previous monographs have presented the theoretical tools which must be used to solve optimization problems. References 1.1, 1.2, and 1.3 contain the developments of the Calculus of Variations, Pontryagin Maximum Principle, and Dynamic Programming, respectively. However, the application of these tools requires a comprehensive knowledge of the formulation of the various types of optimization problems as well as the mathematical theory behind these tools. It is the intent of this section to present the reader with: (1) an understanding of what is to be accomplished in an optimization problem, and (2) a general notion of the formulation of an optimization problem.

The engineer who uses optimization theory is usually faced with the problem of controlling a dynamical system such that a particular mission is accomplished with some measure of performance being extremized (maximum or minimum). As an example, consider the problem of placing a satellite into orbit. The motion of the vehicle is governed by a system of differential equations the initial conditions for which are the values of the system variables as the vehicle leaves the launch pad. The terminal conditions are specified such that the satellite goes into the desired orbit. In order to control this system, the thrust is varied in direction and magnitude; the steering angles and throttle setting are called control variables. The purpose of the optimization problem is to determine time histories of the steering angles and the throttle setting such that the mission is accomplished; and at the same time, some measure of performance is optimized. There are many mission performance criteria that have been developed. Some of the more common ones would be minimum time, minimum fuel, maximum terminal velocity, and many combinations of terminal values of system variables.

Once the performance criterion is selected, it is used with the system equations and the boundary conditions (initial and terminal conditions) to formulate an optimization problem. There are three formulations that are usually discussed in optimization theory, each of which seems to be suited for different types of optimization problems. The Calculus of Variations and the Pontryagin Maximum Principle formulations, in reality two different renditions of the same idea, are more suitable for problems whose system variables are well behaved. Frequently, however, a problem that cannot be solved analytically is encountered. For some of these problems the third formulation, Dynamic Programming, is the only method that can be used to obtain a solution. All three of these formulations will be used in the following sections in order to illustrate their use.

In order to discuss the trajectory optimization problem, a general formulation of the problem must first be written so that the terminology can be established for later development. In this analysis the state variables will be represented by $X_i(t)$, where $i = 1, 2 \text{----} n$, and the state vector will be denoted as

$$\underline{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \cdot \\ \cdot \\ \cdot \\ X_n(t) \end{bmatrix} \tag{1.1}$$

The reader should recognize the state variables as the minimum set of system variables that, if known at some time τ, can completely specify the system behavior for all future time $t > \tau$ in the absence of disturbances. In the case of a space vehicle, the six dimensional vector of position and velocity is a typical state vector.

The next item of interest in a trajectory optimization problem is to specify the control variables of the problem. The control variables are those which can be changed voluntarily to influence the future state of the system. Typical control variables in a space vehicle would be the magnitude and direction of the thrust. The control variables will be denoted by $U_i(t)$ where $i = 1, 2, ----, r;$ and the control vector will be designated as

$$\underline{U}(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \cdot \\ \cdot \\ \cdot \\ U_r(t) \end{bmatrix} \tag{1.2}$$

Each system has a set of equations that describes the dynamics. In the case of a space vehicle, the equations of motion form a set of equations that specifies the system behavior. These equations are called constraint equations and can be written in vector notation as

$$\dot{\underline{X}}(t) = \underline{f}(t, \underline{X}, \underline{U}) \tag{1.3}$$

where $\underline{f}(t, \underline{X}, \underline{U})$ means

$$\underline{f}(t,\underline{X},\underline{U}) = \begin{bmatrix} f_1(t, \underline{X}, \underline{U}) \\ f_2(t, \underline{X}, \underline{U}) \\ \cdot \\ \cdot \\ \cdot \\ f_n(t, \underline{X}, \underline{U}) \end{bmatrix} \tag{1.4}$$

an alternate form of the constraint equations is

$$f_i{}^* = \dot{X}_i(t) - f_i(t, \underline{X}, \underline{U}) \;=\; 0 \qquad i = 1, ---, n \tag{1.5}$$

It should be noted that the dimension of the constraint equation is identical to the dimension of the state vector. Written in this form, the constraint equations are the minimal set of first-order differential equations necessary in order to completely specify the system behavior. Each of the equations is a differential equation for one of the state variables. Hence, it follows that there should be n constraint equations and n state variables.

So far in the statement of the problem, optimization has not been mentioned. The first requirement of an optimization problem is that some quantity exist which can be used to compare the performance of one solution to another. This quantity (called the optimization criteria, pay-off function, cost function, performance index, etc.) must be a scalar; however, the form of this scalar can vary to include functions of the state and control variables, depending on which formulation of the optimization problem is chosen. Three general forms of the optimization problem are the Lagrange problem (optimization of an integral of some function of the state and control variables), the Mayer problem (optimization of a function of the state variables at the final time), and the Bolza problem (optimization of the sum of a function of the final state plus an integral of some other function of the state and control variables). However, since these formulations can be shown to be equivalent, it does not make much difference which one is used; the Mayer form will be used here since this form is popular for trajectory problems. The optimization criterion, which is often referred to as pay-off function, cost function, performance index, etc., will be denoted

$$J = \phi(\underline{X}^f, t^f) \tag{1.6}$$

The object of the optimization problem is to minimize or maximize this function.

As in any trajectory problem, there are certain end conditions that must be met. Usually the end conditions are mixed. That is, some (or perhaps all) of the conditions at the beginning of the mission may be specified, and some (or perhaps all) of the terminal conditions may be specified. It should be noted that all of the end conditions at both ends cannot be specified simultaneously for one mission, since some (at least one) must be left as degrees of freedom for the optimization. (If all are specified, there is nothing to optimize because the pay-off function is a function of the terminal state and terminal time; hence, it is already specified.) The end conditions can be expressed functionally as

$$\underline{X} = \underline{X}^{o} = \begin{bmatrix} X_1{}^{o} \\ X_2{}^{o} \\ \cdot \\ \cdot \\ \cdot \\ X_n{}^{o} \end{bmatrix} \qquad \text{at } t = t^{o} \tag{1.7}$$

and

$$\psi_j(\underline{X}^f, t^f) = 0 \qquad j = 1,---, \quad m \le n \tag{1.8}$$

The former conditions pertain to the beginning of the mission, while the latter conditions pertain to the terminal conditions. Note that there are a total of $2n + 2$ boundary conditions that could conceivably be specified. This number results from the fact that there are n functions of time that could be specified at both $t = t^{o}$ and $t = t^{f}$ in addition to the terminal and initial times. Not more than $2n + 1$ can be specified in an optimization problem.

The only aspect of the problem that remains to be discussed is the control that must be exercised (within certain constraints) in order to optimize the pay-off function. Since most physical problems are limited in the magnitude and nature of control, it is necessary to formulate these additional constraints mathematically and somehow include them in the analytical framework of the solution. This can be accomplished by writing inequality constraint equations such as

$$h_p(t,\underline{X})_{MIN} \le h_p(\underline{U}) \le h_p(t,\underline{X})_{MAX} \qquad p = 1,---, q \tag{1.9}$$

A typical constraint of this form would be a rocket engine where thrust must lie in the range

$$0 \le T \le T_{MAX} \tag{1.10}$$

Unfortunately, an inequality of this form is not as useful as an equality equation, since it is desirable to join this constraint to the other constraints. It is possible, however, to write an equality that contains the same information as the inequality. This is accomplished by first writing Equation (1.9) as

$$g_p(t,\underline{X},\underline{U}) = \left[h_p(t,\underline{X})_{MAX} - h_p(\underline{U})\right]\left[h_p(t,\underline{X})_{MIN} - h_p(\underline{U})\right] \leq 0 \qquad (1.11)$$

and then introducing a new variable, η, that is some measure of the difference between $g_p(t,\underline{X},\underline{U})$ and zero. Thus, Equation (1.11) becomes the equality:

$$g_p(t,\underline{X},\underline{U}) + \eta^2{}_p(t) = 0 \qquad p = 1,\text{---}\ q \qquad (1.12)$$

The only restriction is that $\eta_p(t)$ be a real variable. It is seen that when the control assumes either of its bounded values, both $g_p(t,\underline{X},\underline{U})$ and $\eta^2{}_p(t)$ are equal to zero. For any non-bounded control, g_p is a negative quantity and η^2 is the difference between zero and the value of g_p. This technique provides the capability to include the bounded control constraint in the solution in a manner very similar to the regular system differential constraint equations. The difference in these two types of constraint equations is that the system equations are first-order differential equations, whereas the control equations are only algebraic equations. However, in the analysis they can be treated in a similar manner.

Later in the discussion, three formulations of the optimization problem will be discussed; they are the Calculus of Variations, Pontryagin's Maximum Principle, and Dynamic Programming. In the Calculus of Variations approach, the state and control variables must be continuous. Since this may not be the case in control variables, it is necessary to remove this difficulty by introducing the transformation

$$U = \dot{Z} \qquad (1.13)$$

Now Z can be used in place of U, and Z is continuous even if U is discontinuous.

2.1.1 Calculus of Variations Formulation

The object of the optimization analysis is to determine the set of functions $\underline{X}(t)$ and $\underline{U}(t)$ for which

$$J = \phi(\underline{X}^f, t^f) \qquad (1.14)$$

is a minimum. This optimization must be consistent with the end conditions $\psi_j(\underline{X}^f, t^f)$, the differential constraint equations $f^*{}_i$, and the control constraint equations, $g_p(\dot{z}) + \eta^2{}_p$. To accomplish this solution, these constraints are adjoined to the pay-off function by arbitrary multipliers. Those constraint equations that are not functions of time are simply joined by addition; those constraints that are functions of time are added via an integral. A new pay-off function is thus formed as

$$\bar{J} = \phi(\underline{X}^f, t^f) + \mu_j\ \psi_j\ (\underline{X}^f, t^f) + \int_{t^o}^{t^f} \left\{ p_i \left[\dot{X}_i - f_i(\underline{X}, \underline{\dot{z}}) \right] + \lambda_p \left[g_p(\dot{z}) + \eta^2{}_p \right] \right\} dt \quad (1.15)$$

where μ_j, p_i and λ_p are arbitrary Lagrange multipliers. It should be noted that the pay-off function has the same numerical value since all the terms that were added are zero.

As is described in other monographs (References 1.3 and 1.4), the solution to the problem is obtained by setting the first variation of $\bar{J}$ to zero. The results are the well-known Euler-Lagrange equations, the transversality equations or boundary condition, and the Weierstrass-Erdmann corner conditions. The Weierstrass condition results from the fact that the first variation must be greater than or equal to zero.

(1) Euler-Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{X}_i}\right) - \frac{\partial F}{\partial X_i} = 0 \qquad i = 1, ---, n \qquad (1.16)$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{z}_K}\right) - \frac{\partial F}{\partial z_K} = 0 \qquad K = 1, ---, r \qquad (1.17)$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\eta}_p}\right) - \frac{\partial F}{\partial \eta_p} = 0 \qquad p = 1, ---, q \qquad (1.18)$$

where

$$F = p_i \left[\dot{X}_i - f_i(\underline{X}, \underline{\dot{z}}) \right] + \lambda_p \left[g_p(\dot{z}) + \eta^2{}_p \right]$$

(2) Transversality or Boundary Conditions

$$\frac{\partial F}{\partial X_i} + \mu_j \frac{\partial \psi_j}{\partial X_i} + \frac{\partial \phi}{\partial X_i} = 0 \quad i = 1, ---, n \qquad (1.19)$$

$$\frac{\partial F}{\partial \dot{z}_k} + \mu_j \frac{\partial \psi}{\partial z_k} + \frac{\partial \phi}{\partial z_k} = 0 \qquad k = 1, \ldots, r \tag{1.20}$$

$$\frac{\partial F}{\partial \dot{\eta}_p} + \mu_j \frac{\partial \psi}{\partial \eta_p} + \frac{\partial \phi}{\partial \eta_p} = 0 \qquad p = 1, \ldots, q \tag{1.21}$$

$$F - \frac{\partial F}{\partial \dot{x}_i} \dot{x}_i - \frac{\partial F}{\partial \dot{z}_k} \dot{z}_k - \frac{\partial F}{\partial \dot{\eta}_p} \dot{\eta}_p - \mu_j \frac{\partial \psi_j}{\partial t} - \frac{\partial \phi}{\partial t} = 0 \tag{1.22}$$

(3) <u>Weierstrass-Erdmann Corner Conditions</u>

$$\frac{\partial F^{(-)}}{\partial \dot{x}_i} = \frac{\partial F^{(+)}}{\partial \dot{x}_i} \tag{1.23}$$

$$\frac{\partial F^{(-)}}{\partial \dot{z}_k} = \frac{\partial F^{(+)}}{\partial \dot{z}_k} \tag{1.24}$$

$$\frac{\partial F^{(-)}}{\partial \dot{\eta}_p} = \frac{\partial F^{(+)}}{\partial \dot{\eta}_p} \tag{1.25}$$

$$F^{(-)} - \frac{\partial F^{(-)}}{\partial \dot{x}_i} \dot{x}_i^{(-)} - \frac{\partial F}{\partial \dot{z}_k} \dot{z}_k^{(-)} - \frac{\partial F^{(-)}}{\partial \dot{\eta}_p} \dot{\eta}_p^{(-)} = F^{(+)} - \frac{\partial F^{(+)}}{\partial \dot{x}_i} \dot{x}_i^{(+)} - \frac{\partial F^{(+)}}{\partial \dot{z}_k} \dot{z}_k^{(+)} - \frac{\partial F^{(+)}}{\partial \dot{\eta}_p} \dot{\eta}_p^{(+)} \tag{1.26}$$

(4) <u>Weierstrass Condition</u>

$$F(\dot{\underline{X}}, \dot{\underline{Z}}, \eta) - F(\dot{\underline{x}}, \dot{\underline{z}}, \eta) - (\dot{X}_i - \dot{x}_i) \frac{\partial F}{\partial \dot{x}_i} - (\dot{Z}_k - \dot{z}_k) \frac{\partial F}{\partial \dot{z}_k} \geq 0 \tag{1.27}$$

where lower case variables denote the optimal function and the capitals denote any other function that satisfy the differential constraint equations and the control constraint equations.

In subsequent sections [Sections (2.2.1) and (2.2.3)] some simple optimization problems are solved using this formulation. In these problems, a closed-form solution is obtained because the assumptions that are made reduce the complexity of the problem. In general, the trajectory optimization problem is highly nonlinear, and closed-form solutions to the equations are not possible. Thus, the solutions must be obtained by numerical techniques such as the gradient (or steepest descent), quasilinearization, or neighboring extremal formulations. Since these techniques are discussed in detail in other monographs [References 1.3 and 1.4], they will not be pursued in detail here. However, some general remarks on the techniques are in order.

The first step in the numerical solution is to assume some time histories for the solution that satisfy some, but not all, of the governing equations. In the case of the gradient technique, a control $\underline{U}$ (t) which permits the state equations to satisfy the initial and terminal state constraints is selected. In the neighboring extremal technique, control $\underline{U}$ (t) and state X(t) histories that satisfy the differential constraint equations and the optimization condition are selected. In quasilinearization a starting solution that satisfies the boundary conditions is selected. All of the techniques then use first-order (in the gradient method) or second-order (neighboring extremal or qualilinearization) theory to determine corrections to the initial guesses of the solutions. The process continues in an iterative fashion until the pay-off function converges to its minima within an acceptable tolerance. The complete analysis of all these techniques is covered in Reference 1.5.

2.1.2 The Pontryagin Maximum Principle Formulation

The very same constraint equations can be formulated in a slightly different manner using the Pontryagin Maximum Principle. However, although the two methods are equivalent, the Maximum Principle formulation sometimes permits more insight in the selection of the optimum control. Later in the discussion, a comparison between the results of both formulations will be made.

When the Maximum Principle is used to formulate a problem, the hamiltonian is needed; this function is obtained by forming the sum

$$H = \sum_{i=1}^{N} p_i f_i \tag{1.28}$$

where p_i is some arbitrary multiplier. (It can be shown that the p_i's which are called co-state variables, are equivalent to the p_i's in the Calculus of Variations formulation.) The next step is to find the differential co-state equations. These are

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -p_j \frac{\partial f_j}{\partial x_i} \tag{1.29}$$

The following table relates the various constraint equations for the Calculus of Variation formulation.

CONDITION	COV	PMP
Differential Constraint Equation	$\dot{x}_i = f_i(\underline{x}, \underline{u})$	$\dot{x}_i = f_i(\underline{x}, \underline{u})$
Control Constraint Equation	$g_p(\dot{\underline{z}}) + \eta_p^2 = 0$ $\dot{\underline{z}} = \underline{u}$	$g_p(\underline{u}) + \eta_p^2 = 0$
Initial Conditions	$\underline{x}(t) = \underline{x}(0)$ at $t = t^0$	$\underline{x}(t) = \underline{x}(0)$ at $t = t^0$
Terminal Conditions	$\psi_j(\underline{x}^f, t^f) = 0$	$\psi_j(\underline{x}^f, t^f) = 0$
Payoff Function	$J = \phi(\underline{x}^f, t^f) = MIN$	$J = \phi(\underline{x}^f, t^f) = MIN$
Co-state Equations	$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) - \frac{\partial F}{\partial x_i} = 0$	$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -p_j \frac{\partial f_j}{\partial x_i}$
Co-state Boundary Conditions (Transversality)	$\frac{\partial F}{\partial \dot{x}_i} + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0$ $F - \frac{\partial F}{\partial \dot{x}_i}\dot{x}_i + \mu_j \frac{\partial \psi_j}{\partial t} = 0$	$p_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0$ $H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t}$
Corner Conditions	$\frac{\partial F^{(-)}}{\partial \dot{x}_i} = \frac{\partial F^{(+)}}{\partial \dot{x}_i}$ $F^{(-)} - \frac{\partial F^{(-)}}{\partial \dot{x}_i}\dot{x}_i^{(-)} = F^{(+)} - \frac{\partial F^{(+)}}{\partial \dot{x}_i}\dot{x}_i(t)$	$p_i^{(-)} = p_i^{(+)}$ $H(\underline{p}^{(-)}, x, \underline{u}^{(-)}) = H(\underline{p}^{(+)}, x, u^{(+)})$

A COMPARISON OF FORMULATIONS

The co-state differential equations yield the same results as the application of the Euler-Lagrange equation. Thus, the total number of differential equations involved in the solution is $2n$, n of which are the original differential constraint equations and n are the co-state differential equations.

As was the case in the Calculus of Variations formulation, the boundary conditions are of the mixed type; that is, some are specified at the beginning of the mission and others are specified at the end of the mission. Therefore, a numerical solution must usually be performed to overcome this difficulty. This solution must satisfy the co-state boundary conditions

$$p_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0 \qquad at \quad t = t_f \tag{1.30}$$

The initial and terminal values of the state variables specified (The initial state variables are merely the components of the initial state vector and final state variables are specified in $\psi_j(\underline{x}^f, t^f) = 0$) and the terminal value of the hamiltonian

$$H = \frac{\partial \phi}{\partial t} + \mu_j \frac{\partial \psi_j}{\partial t} \tag{1.31}$$

It should be noted that the end conditions in Equation (1.30) are merely the modified form of the transversality conditions suitable for use with this formulation.

The corner condition and the Weierstrass condition are the only remaining conditions to be discussed in this formulation. The corner conditions simply require the co-state variables and the hamiltonian be continuous at the corner point, i.e.,

$$p_i^{(-)} = p_i^{(+)} \tag{1.32}$$

$$H\left[\underline{p}^{(-)}, \underline{x}, \underline{u}^{(-)}\right] = H\left[\underline{p}^{(+)}, \underline{x}, \underline{u}^{(+)}\right] \tag{1.33}$$

and the Weierstrass condition requires that any control other than the optimal yields a hamiltonian that is smaller than that obtained by using the optimal control, i.e.,

$$H(\underline{x}, \underline{p}, \underline{u}_o) \geq H(\underline{x}, \underline{p}, \hat{\underline{u}}) \tag{1.34}$$

where $\underline{U}_o$ is the optimal control and $\underline{U}$ is any other control.

CONDITION	COV	PMP
Weierstrass Condition	$F(X) - F(x) - (X_i - x_i)\dfrac{\partial F}{\partial x_i} \geq 0$	$H(\underline{x}, \underline{p}, \underline{u}_o) \geq H(\underline{x}, \underline{p}, \hat{\underline{u}})$ U_o is the optimal control

A COMPARISON OF FORMULATIONS (continued)

A deeper investigation of the similarities between the Calculus of Variations and the Pontryagin Maximum Principle indicates that the two formulations are more than similar; they are different renditions of the same idea. That is, the Euler-Lagrange equations in the Calculus of Variations yield the same set of equations as the co-state differential equations in the Maximum Principle formulation. Furthermore, if the augmented function, F, is independent of t, the first integral of the Euler-Lagrange equation is the hamiltonian. The transersality equations and the corner conditions can be seen to be equivalent from the previous chart when it is realized that

$$\frac{\partial F}{\partial \dot{x}_i} = p_i \qquad (1.35)$$

2.1.3 Dynamic Programming Formulation

In some cases, analytical solutions to optimization problems are not possible because of irregularities in the behavior of state variables or of certain constraints on the control or state variables. In such cases, the techniques of Dynamic Programming may be of use in obtaining a solution. Dynamic Programming is implemented by first dividing the state space into many parts. The payoff function is then computed for each point in the state space. By using the Principle of Optimality, the optimum value of the payoff function and the optimum decision sequence is recorded for each point in the state space. The optimum trajectory is then found by following the decision sequence from the beginning of the mission to the end of the mission.

It should be noted that such a procedure quickly saturates the core storage capabilities of most modern computers even for modest trajectory problems. Thus, numerical techniques and sophisticated coding logic must be used to overcome this difficulty. This disadvantage is probably one of the most significant reasons for the stunted growth of Dynamic Programming in trajectory optimization.

Dynamic Programming should not be interpreted as an entirely different technique from the Calculus of Variations or the Maximum Principle. Basically, all three use the same concepts with different implementations. In fact, the equations developed from variational theory can be obtained by using a limiting process on the equation for the Principle of Optimality. Hence, Dynamic Programming can be interpreted as a discretized version of variational theory.

2.2 Boost Problems

Previous pages have summarized some of the results prepared earlier; the problem as outlined now requires that the material be applied. To this end, several problems related to one of the more critical aerospace applications, boost of rocket propelled vehicles, have been selected for review.

First, a simple vertical shoot problem with a closed form solution is considered. Here, the form of the solution is obtained to within some unknown constants. Subsequent problems (Sections 2.2.2, 2.2.3, and 2.2.4) have solutions that are not closed form in nature since iteration is needed in order to obtain the solution, even though the form of the solution may be known. Finally, the full non-linear boost problem is considered. This problem is one in which the entire solution must be found by numerical techniques such as the gradient method or quasilinearization. The section ends with the determination of optimum staging of boost vehicles.

2.2.1 The Vertical Shoot Problem

In order to demonstrate the application of the theory that has been presented in previous monographs and reviewed in Section 2.1, a very simple trajectory problem will be analyzed. Specifically, the problem to be considered is that of determining the optimum burning program for a rocket in vertical flight such that a maximum altitude is attained. The following simplifying assumptions will be made so that the basic concepts can be stressed:

1. the Earth is flat
2. the gravitational acceleration is constant throughout the entire trajectory
3. the flight trajectory is in a vacuum
4. the flight trajectory is vertical
5. the thrust is tangent to the flight path
6. the equivalent exit velocity of the rocket engine is constant
7. the engine is throttleable, i.e., it has a variable mass flow rate between an upper and lower limit

Actually, the reader might suspect the answer to this problem intuitively to be to burn at the maximum thrust until all the propellant is consumed and coast for the remainder of the flight. However, because the vehicle is throttleable, it is not obvious that a full throttle policy is <u>the</u> optimum policy. The following analysis will show that the optimum policy is indeed the "full-on" or "full-off" policy.

The equations of motion for the rocket are

$$V = \dot{h} \tag{2.1}$$

$$\dot{V} = cu/m - g \tag{2.2}$$

$$\dot{m} = -u \tag{2.3}$$

where V = velocity

h = altitude

c = the equivalent exhaust velocity

u = the propellant mass flow

m = the mass of the vehicle

g = the gravitational acceleration

Thus,

$$f_1^* \equiv \dot{h} - V = 0 \tag{2.4}$$

$$f_2^* \equiv \dot{V} + g - cu/m = 0 \tag{2.5}$$

$$f_3^* \equiv \dot{m} + U = 0 \tag{2.6}$$

Now noting that the mass flow rate of the engine has an upper limit, U_{max}, and a lower limit of zero, i.e.,

$$0 \leq U \leq U_{MAX} \tag{2.7}$$

requires that an inequality constraint be introduced. This objective can be accomplished by rewriting the inequality as an equality by the introduction of a variable α such that

$$\alpha^2 = U(U_{MAX} - U) \tag{2.8}$$

where α is a real number as long as $0 \leq U \leq U_{max}$. Expressed in the zero form, Equation (2.8) becomes a fourth constraint equation

$$f_4^* \equiv U(U_{MAX} - U) - \alpha^2 = 0 \tag{2.9}$$

The system of equations being considered is thus a set of four equations f_1^*, f_2^*, f_3^*, f_4^*, three of which are differential equations involving five dependent variables, h, V, m, U, α and one independent variable, t. At this point, it is advantageous to investigate some properties of the problem so that it can be recognized as a legitimate variational problem. First, it should be pointed out that there are four constraint equations and five dependent variables, thus leaving one degree of freedom for control. This variable is the one upon which some optimum requirement can be imposed. Second, since there are three differential constraint equations, a total of eight ($2n+2$ where $n = 3$) separate boundary conditions exist. All eight boundary conditions cannot be specified, since some freedom must be left for the variables to be optimized. In this particular problem, the variable to be optimized is the final altitude, h^f . The boundary conditions that are specified are t^o , h^o , V^o , m^o , V^f , and m^f . So far only seven boundary conditions have been mentioned. The remaining one is the terminal time. Since no restriction has been mentioned for time, it will be left unspecified as is the final altitude. All eight boundary conditions are now accounted for.

The problem is to find the time history of the functions $h(t)$, $V(t)$, $m(t)$, $U(t)$, and $\alpha(t)$, which are consistent with the constraint equations and the boundary conditions, and at the same time minimize

$$J = \phi(x^f, t^f) = -h^f \tag{2.10}$$

(Maximizing a function is equivalent to minimizing the negative of the function. In this case, h is maximized by minimizing -h.)

This problem will be solved by using the Calculus of Variations. First, an augmented function, F, is formed with the use of the Lagrange multipliers, p_i , to join the constraint equations

$$F = \sum_{i=1}^{4} p_i f_i^* = p_1(\dot{h} - V) + p_2(\dot{V} + g - cu/m) + p_3(\dot{m} + u) + p_4\left[u(u_{MAX} - u) - \alpha^2\right] \tag{2.11}$$

Then the Euler-Lagrange conditions applied yield five equations, one for each of the dependent variables. They are

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{h}}\right) - \frac{\partial F}{\partial h} = 0 \tag{2.12}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{V}}\right) - \frac{\partial F}{\partial V} = 0 \tag{2.13}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{m}}\right) - \frac{\partial F}{\partial m} = 0 \tag{2.14}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right) - \frac{\partial F}{\partial u} = 0 \tag{2.15}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\alpha}}\right) - \frac{\partial F}{\partial \alpha} = 0 \tag{2.16}$$

These equations reduce to the following

$$\dot{p}_1 = 0 \tag{2.17}$$

$$\dot{p}_2 = -p_1 \tag{2.18}$$

$$\dot{p}_3 = p_2 \frac{cu}{m^2} \tag{2.19}$$

$$p_3 - \frac{c}{m} p_2 + p_4 (u_{MAX} - 2u) = 0 \tag{2.20}$$

$$p_4 \alpha = 0 \tag{2.21}$$

In Reference 1.1, Page 102, it is shown that if in the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}_i}\right) - \frac{\partial F}{\partial y_i} = 0 \quad (i = 1, \ldots, n) \tag{2.22}$$

the function F is formally independent of t, (i.e., $\partial F/\partial t = 0$), then the equation possess the first integral

$$-F + \sum_{i=1}^{N} \frac{\partial F}{\partial \dot{y}_i} \dot{y}_i = CONST. = C \tag{2.23}$$

For the problem at hand, this integral is

$$-F + \frac{\partial F}{\partial \dot{h}} \dot{h} + \frac{\partial F}{\partial \dot{V}} \dot{V} + \frac{\partial F}{\partial \dot{m}} \dot{m} + \frac{\partial F}{\partial \dot{u}} \dot{u} + \frac{\partial F}{\partial \dot{\alpha}} \dot{\alpha} = C \tag{2.24}$$

Thus, combining this equation with Equation (1.11) provides

$$p_1 V - p_2 g + KU = C \tag{2.25}$$

where $K = p_2 \left(\frac{c}{m}\right) - p_3$.

It was previously stated that six of the eight boundary conditions were specified and that the final altitude and final time were unspecified, the final altitude being that parameter to be optimized. In order to solve the system of equations, however, some more information on the two unspecified boundary conditions is needed. This additional information is obtained from the application of the transversality conditions. In principle, this is equivalent to solving the simple variational problem in which the boundary is located on some curve but the particular point is not specified. It is the transversality condition that gives the remaining boundary information that is necessary in order to obtain an explicit solution. The transversality condition is

$$\left[d\phi + \left(F - \sum_{k=1}^{N} \frac{\partial F}{\partial \dot{y}_k} \dot{y}_k \right) dt + \sum_{k=1}^{N} \frac{\partial F}{\partial \dot{y}_k} dy_k \right]_0^f = 0 \tag{2.26}$$

The second term is seen to be the first integral of Equation (2.23). Hence, Equation (2.26) becomes

$$\left[d\phi - C\,dt + \sum_{k=1}^{N} \frac{\partial F}{\partial \dot{y}_k} dy_k \right]_0^f = 0 \tag{2.27}$$

Substitution of this relation into Equation (2.27) yields

$$\left[-dh - \underbrace{(p_1 V - p_2 g + Ku)}_{C} dt + p_1\,dh + p_2\,dV + p_3\,dm \right]_0^f = 0 \tag{2.28}$$

or

$$\left[-C\,dt + (p_1 - 1)\,dh + p_2\,dV + p_3\,dm \right]_0^f = 0 \tag{2.29}$$

But, since the velocity and mass are specified, dV = 0 and dm = 0, the remaining terms must be

$$C = 0$$

$$p_1 - 1 = 0 \quad \text{OR} \quad p_1 = 1 \tag{2.30}$$

Now, the condition to be satisfied along the extremal can be found by employing the definition of K and its first derivative

$$\dot{K} = \frac{-c\,p_2}{m^2} \dot{m} + \frac{c}{m} \dot{p}_2 - \dot{p}_3 \tag{2.31}$$

But

$$\dot{p}_2 = -1 \quad \text{and} \quad \dot{p}_3 = -p_2 \frac{cu}{m^2}$$

so

$$\dot{K} = \frac{-p_2 c}{m^2} (\dot{m} + u) - \frac{c}{m} \tag{2.32}$$

However, the constraint equation for the thrust is

$$\dot{m} + u = 0 \tag{2.33}$$

so finally

$$\dot{K} = -\frac{c}{m} \tag{2.34}$$

The equations that have been derived so far must now be interpreted. First, Equation (2.21) states that the product of p_4 and α is always zero. Hence, at any one time either p_4 or α or both must be zero. Consider the case when $\alpha = 0$ first. For this case, constraint Equation (2.9) (f_4*) restricts the value of U to be

$$U = 0 \qquad \text{or} \qquad U = U_{max} \tag{2.35}$$

Next, consider the case of $p_4 = 0$. Equation (2.20) then becomes

$$p_3 - \frac{c}{m} p_2 = 0 \tag{2.36}$$

or $K = 0$ for all time when $p_4 = 0$, which implies that $K = 0$. But Equation (2.34) cannot be satisfied if $K = 0$, so there are no subarcs on which $p_4 = 0$. It can be concluded, therefore, that there can only be two kinds of subarcs: (1) those on which the thrust is off, and (2) those on which the thrust is on full throttle. There can be no subarcs on which the thrust is variable between the limits stated.

So far, the fact that the thrust must be either full on or full off has been established; but no information involving when it should be on or off has been determined. This information can be partially obtained by employing some additional conditions that must be satisfied and by using continuity information of various variables. Some insight to the allowable times for a thrust change can be obtained by examining Equation (2.25). First, it must be realized that the variables p_1, p_2, and p_3 are to satisfy differential Equations (2.17), (2.18), and (2.19). Hence, they must be continuous under normal circumstances. In addition, the constant C must retain the same value on all subarcs of the solution. But, the velocity must also be continuous; thus, $K = 0$ is the only choice for a discontinuity in U, the mass flow. This conclusion results from the fact that all terms of Equation (2.25) are continuous, except the third term, so that the only way that U can change from $U = 0$ to $U = U_{max}$ (or vice-versa) is for K to be zero at the time of the change.

The Euler-Lagrange equations is a necessary condition for an extremum; however, the Legendre-Clebsch condition is also necessary. The application of this condition yields the relationship

$$p_4 \left[(\delta\alpha)^2 + (\delta u)^2 \right] \geq 0 \tag{2.37}$$

From Equation (2.20) and the definition of K, Equation (2.37) becomes

$$\frac{2K}{2U-U_{MAX}}\left[(\delta\alpha)^2+(\delta U)^2\right]\geq 0 \tag{2.38}$$

Now, it should be recalled that variations in α and U must still be consistent with constraint equation $f_4{}^*$. A relation between K and U can be obtained by examining Equation (2.38) for the two conditions ($U = U_{max}$ and $U = 0$). It is seen that if $U = U_{max}$, it is only necessary for $K \geq 0$ for Equation (2.38) to be satisfied. Also, if $U = 0$, Equation (2.38) is satisfied for $K \leq 0$. Hence, a parameter that can be used to determine the thrust has been found. K can be called a switching function, for it tells whether the control should be "on" or "off" depending on the value of K. In this case, K is positive for the initial part of the flight when the vehicle is flown with the engine on full, zero for the corner point when the switching occurs, and negative for the coasting phase. Thus, if an analytical expression for K can be obtained for the engire trajectory, some insight to the burning policy can be obtained. First, consider the case when $K > 0$. It has already been established that when $K > 0$, then $U = U_{max}$ so that m is a time varying quantity in the differential equation

$$\dot{K} = -\frac{c}{m} \tag{2.39}$$

Thus, the analytical expression for K can be determined by integration

$$\dot{K} = \frac{dK}{dm}\cdot\frac{dm}{dt} = -\frac{c}{m} \tag{2.40}$$

where $\dfrac{dm}{dt} = -U_{MAX}$

The result is

$$K = \frac{c}{U_{MAX}}\ln m + c_1 \tag{2.41}$$

where c_1 is a constant of integration which can be determined by specifying that the mass is equal to some m_c when switching occurs at $K = 0$. Hence,

$$c_1 = \frac{-c}{U_{MAX}}\ln mc \tag{2.42}$$

or

$$K = \frac{c}{U_{MAX}}\ln\left(\frac{m}{m_c}\right) \tag{2.43}$$

For the case when K $<$ 0, U = 0. Hence, in the integration of

$$\dot{K} = - \frac{c}{m_c} \tag{2.44}$$

the m is now a constant so

$$K = \frac{-c}{m_c} t + c_2 \tag{2.45}$$

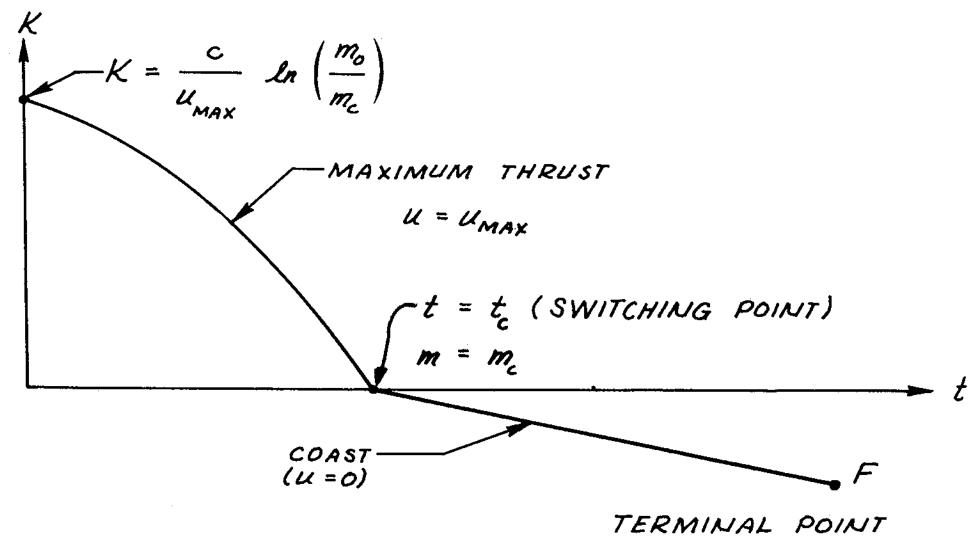
The constant of integration, c_2, can be obtained by recalling that K = 0 at $t = t_c$ so

$$c_2 = \frac{c\, t_c}{m_c} \tag{2.46}$$

and finally,

$$K = -\frac{c}{m} t + \frac{c}{m} t_c = \frac{c}{m}(t_c - t) \qquad (K \leq 0) \tag{2.47}$$

A plot of a typical K is shown in the following sketch (Reference 1.1).



To summarize the solution to the vertical shoot problem, the only allowed values of thrust that could be used were $U = 0$ and $U = U_{max}$; the decision as to which control to use is determined by the switching function, K (K is a variable that starts at some $K_o = c/U_{MAX} \ln (m_o/m_c)$ and varies logarithmically as long as the rocket burns.); as soon as $K = 0$, or $m = m_c$ (a specified mass), the thrust is terminated and a coast trajectory is used for the remaining flight; finally, the K function varies linearly during this final phase and has a negative slope. Therefore, K never reaches $K = 0$ again. Hence, thrust is never reinitiated.

Acknowledgment is given to Angelo Miele whose contributions to trajectory optimization were of assistance in the preparation of this section.

2.2.2 The Goddard Problem With Bounded Thrust

Section 2.2.1 presented the vertical shoot problem in which it was desired to maximize the altitude of a rocket vehicle with a bounded control. The analysis assumed that no drag acted on the vehicle and as a result there were no singular arcs in the solution. That is, the control was of the "bang-bang" type so that it was either "full on" or "full off." This behavior was a direct result of the fact that the switching function was never equal to zero for a non-zero time interval.

However, under certain circumstances singular arcs (that is, an extremal arc on which the control takes some intermediate value) can occur in optimization problems. The Goddard problem is one instance where singular arcs can occur. This section will present the Goddard problem and the basic solution.

The Goddard problem differs from the vertical shoot problem in Section 2.2.1 in that it considers the drag that acts on the vehicle as it passes through the atmosphere. It is this added consideration that enables the singular arc to occur. For this problem, the system constraint equations are

$$\dot{h} = V \tag{2.48}$$

$$\dot{V} = -g - \frac{D(h, V)}{m} + \frac{cu}{m} \tag{2.49}$$

$$\dot{m} = -u \tag{2.50}$$

$$u(u_{MAX} - u) - \alpha^2 = 0 \tag{2.51}$$

where D(h,V) is a general drag function assumed to act opposite in direction to the velocity vector. Thus, the hamiltonian is

$$H = p_h + p_V\left(-g - \frac{D(h, V)}{m} + \frac{cu}{m}\right) + p_m(-u) \tag{2.52}$$

Finally, the control constraint is joined to the hamiltonian by an arbitrary multiplier, λ , to form the augmented hamiltonian, $\bar{H}$.

$$\bar{H} = H + \lambda \left[U(U_{MAX} - U) - \alpha^2 \right] \tag{2.53}$$

Now, the control that maximizes the hamiltonian is found by setting the partial derivatives of $\bar{H}$ with respect to U and α equal to zero.

$$\frac{\partial \bar{H}}{\partial U} = -p_m + p_v \frac{c}{m} + \lambda \left[U_{MAX} - 2U \right] = 0 \tag{2.54}$$

$$\frac{\partial \bar{H}}{\partial \alpha} = \lambda(-2\alpha) = 0 \tag{2.55}$$

From Equation (2.55), it is seen that either $\lambda = 0$, $\alpha = 0$ or $\lambda = \alpha = 0$. When $\alpha = 0$, the "bang-bang" control policy is the optimal control; that is, there are only two possible choices for U and they are $U = 0$ and $U = U_{max}$. This solution can be seen by writing the augmented hamiltonian as

$$\bar{H} = U(p_v \frac{c}{m} - p_m) + \lambda \left[U(U_{MAX} - U) - \alpha^2 \right] \tag{2.56}$$

The hamiltonian is maximized if U is chosen to be U_{max} when (p_v $c/m - p_m$) is greater than zero, and if U is chosen to be zero when (p_v $c/m - p_m$) is less than zero. Hence, if $\alpha = 0$, the standard "bang-bang" control policy is optimal.

Now, consider the case where $\lambda = 0$. Equation (2.54) can be satisfied under this condition only if

$$-p_m + p_v \frac{c}{m} = 0 \tag{2.57}$$

It should be recognized that this switching function is a particular case for which the value is identically equal to zero. The control for a non-zero valued switching function has already been established. The remainder of this section will involve the determination of the correct control for the singular case when the switching function is identically zero.

If Equation (2.57) is to be satisfied for the entire duration of the singular arc, its first and second derivatives must also be zero. And, if the values for the derivatives of the co-state variables are combined with the first and second derivatives of the switching function, a value for the control variable can be obtained. Proceeding formally, the first derivative of the switching function with respect to time is

$$-\dot{p}_m + \dot{p}_V \frac{c}{m} + p_V c \left(-\frac{1}{m^2}\right)\dot{m} = 0 \tag{2.58}$$

Using Equation (2.50) and the definition of the co-state variables,

$$\dot{p}_m = -\frac{\partial H}{\partial m} = p_V \left(\frac{cU}{m^2} + \frac{D(h,V)}{m^2}\right) \tag{2.59}$$

$$\dot{p}_V = -\frac{\partial H}{\partial V} = -p_h + \frac{p_V}{m} \frac{\partial D}{\partial V} \tag{2.60}$$

Thus, the first derivative of the switching function reduces to

$$-p_V \frac{D(h,V)}{m^2} - p_h \frac{c}{m} + \frac{p_V c}{m^2} \frac{\partial D}{\partial V} = 0 \tag{2.61}$$

Note that there are no terms in Equation (2.61) that contain U, the control variable. However, such terms will appear if Equation (2.61) is differentiated one more time:

$$\begin{aligned} &\frac{-D}{m^2} \dot{p}_V - p_V D \frac{(-2)}{m^3} \dot{m} - \frac{c}{m} \dot{p}_h - p_h \frac{c(-1)}{m^2} \dot{m} \\ &+ \frac{c}{m^2} \frac{\partial D}{\partial V} \dot{p}_V + p_V c \frac{\partial D}{\partial V} \frac{(-2)}{m^3} \dot{m} = 0 \end{aligned} \tag{2.62}$$

Using Equation (2.60) for $\dot{p}_V$ and

$$\dot{p}_h = -\frac{\partial H}{\partial h} = -p_V \frac{\partial D}{\partial h} \tag{2.63}$$

for $\dot{p}_h$, Equation (2.62) becomes

$$\begin{aligned} &\frac{-D}{m^2}\left[-p_h + \frac{p_V}{m} \frac{\partial D}{\partial V}\right] - \frac{2p_V DU}{m^3} + \frac{c\,p_V}{m^2} \frac{\partial D}{\partial h} - \frac{p_h cU}{m^2} \\ &+ \frac{c}{m^2} \frac{\partial D}{\partial V}\left[-p_h + \frac{p_V}{m} \frac{\partial D}{\partial V}\right] + \frac{2p_V cU}{m^3} \frac{\partial D}{\partial V} = 0 \end{aligned} \tag{2.64}$$

But, this equation can be solved explicitly for U. The result is

$$u = \frac{-\frac{D p_h}{m^2} + \frac{D p_v}{m^3}\frac{\partial D}{\partial V} - \frac{C p_v}{m^2}\frac{\partial D}{\partial h} + \frac{p_h C}{m^2}\frac{\partial D}{\partial V} - \frac{C p_v}{m^3}\left(\frac{\partial D}{\partial V}\right)^2}{\frac{2 p_v C}{m^3}\frac{\partial D}{\partial V} - \frac{p_h C}{m^2} - \frac{2 p_v D}{m^3}} \tag{2.65}$$

Hence, the optimal control during the singular arc can be found from Equation (2.65) providing expressions for, or values of, D(h, V) and its derivatives with respect to V and h are known.

As an example, consider a vehicle of reference area S. If the drag coefficient (which is a function of the mach number) is assumed to be constant during the singular arc period, the drag is

$$D = \frac{1}{2}\rho V^2 S\, C_D \tag{2.66}$$

where ρ = density of the atmosphere. Thus, for the purposes of the example, if an exponential model for the atmosphere is used

$$\rho = \rho_0 e^{-kh}$$

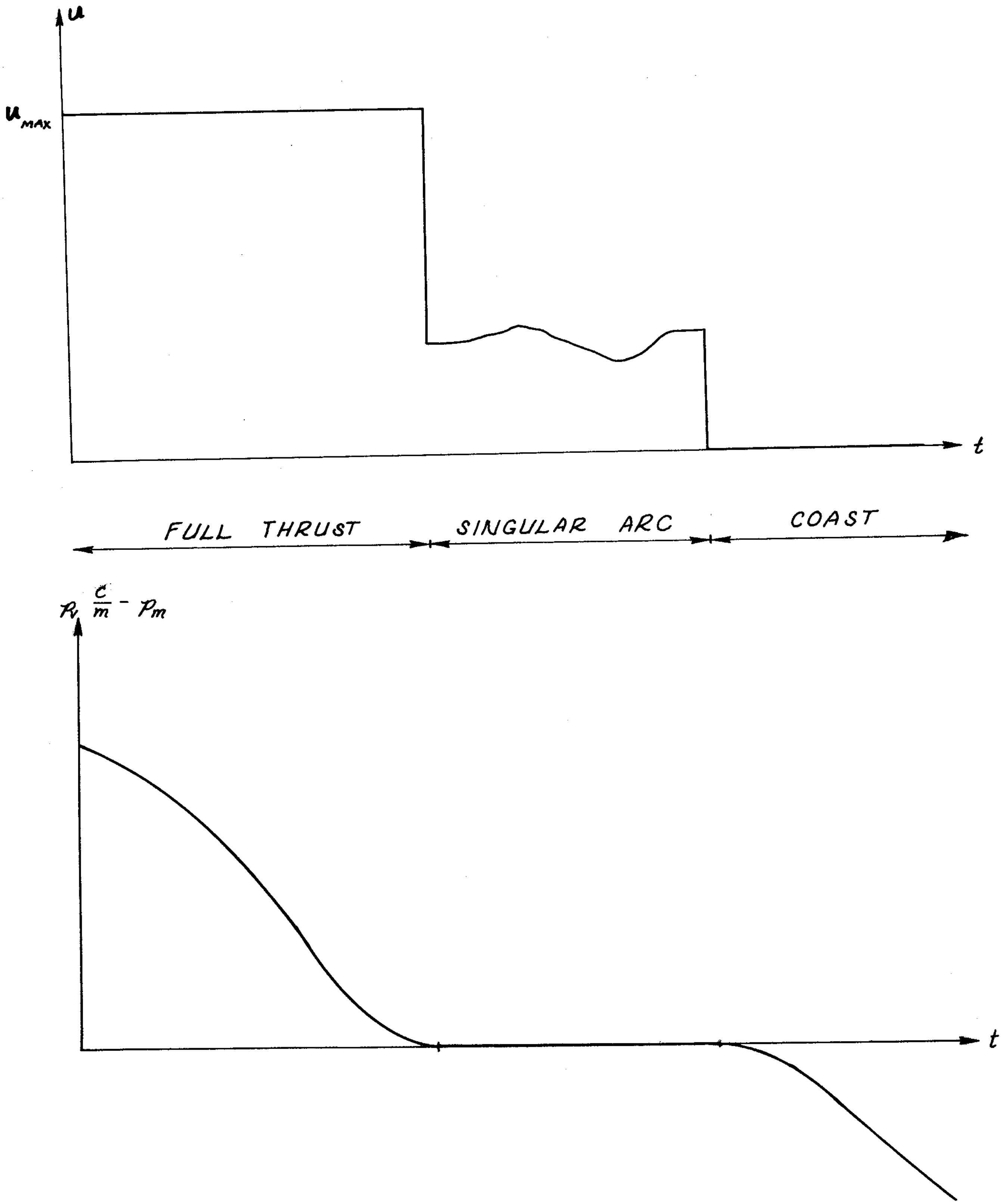
the drag expression becomes

$$D = \frac{1}{2}\rho_0 e^{-kh} V^2 S\, C_D \tag{2.67}$$

The expression for the control during the singular arc period can now be found by the substitution of Equation (2.67) into Equation (2.65). The result is

$$u = \frac{-p_h + C k p_v + \frac{2 p_h C}{V} + \left(\frac{p_v}{mV} - \frac{2C p_v}{m V^2}\right)\rho_0 e^{-kh} V^2 S C_D}{\frac{4 p_v C}{mV} - \frac{2 p_h C e^{-kh}}{\rho_0 V^2 S C_D} - \frac{2 p_v}{m}} \tag{2.68}$$

The following sketches show typical time histories of the control and the switching function.



u
u_{MAX}
t
FULL THRUST
SINGULAR ARC
COAST
$p_v \frac{c}{m} - p_m$
t

2.2.3 The Two-Dimensional Steered Rocket Problem

The next problem to be considered relaxes two of the assumptions that were made in the previous "vertical shoot" problem. Specifically, the vehicle is now permitted to move in a plane and the thrust is controlled in its magnitude and its direction; i.e., thrust is no longer restricted to be tangent to the flight path. The other assumptions of the "vertical shoot" problem still apply, namely

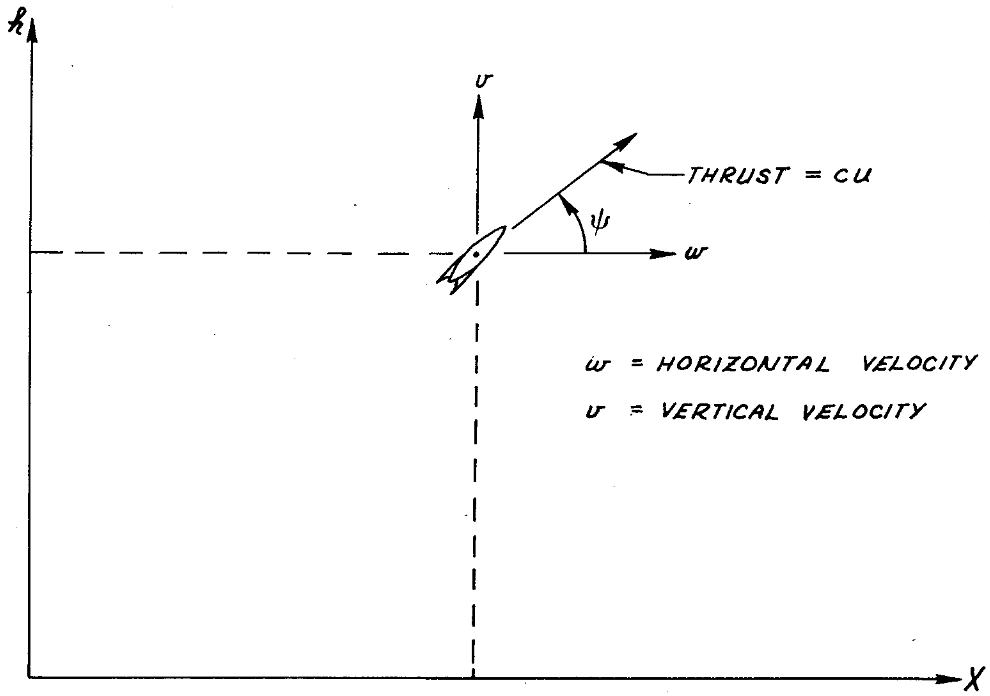
1. The earth is flat and the acceleration of gravity is constant throughout the trajectory

2. The trajectory is in a vacuum

3. The equivalent exit velocity of the rocket is constant

4. The mass flow rate of the rocket has an upper and lower limit

The following sketch shows the problem geometry.



The equations of motion of this vehicle are

$$\dot{X} = w \tag{2.69}$$

$$\dot{h} = v \tag{2.70}$$

$$\dot{u} = \frac{cu}{m} \cos \psi \tag{2.71}$$

$$\dot{v} = \frac{cu}{m} \sin \psi - g \tag{2.72}$$

while the thrust equations are

$$\dot{m} = -u \tag{2.73}$$

and

$$u(u_{MAX} - u) - \alpha^2 = 0 \tag{2.74}$$

as in the last problem. Thus,

$$f_1^* = \dot{X} - w = 0 \tag{2.75}$$

$$f_2^* = \dot{h} - v = 0 \tag{2.76}$$

$$f_3^* = \dot{u} - \frac{cu}{m} \cos \psi = 0 \tag{2.77}$$

$$f_4^* = \dot{v} + g - \frac{cu}{m} \sin \psi = 0 \tag{2.78}$$

$$f_5^* = \dot{m} + u = 0 \tag{2.79}$$

$$f_6^* = u(u_{MAX} - u) - \alpha = 0 \tag{2.80}$$

This system of equations differs from the set for the previous problem in two ways: (1) three more independent variables appear $[X(t),\ w(t),\ \psi(t)]$ and (2) there are two degrees of freedom for control instead of one $[u(t)$ and $\psi(t)]$. Thus, the optimization problem is to find the class of functions that are consistent with the constraint equations (f_i^*), the end conditions and at the same time minimize

$$J = \phi(\underline{x}^f, t^f) \tag{2.81}$$

where

$$\phi = \phi(t^f, x^f, h^f, u^f, v^f, m^f)$$

This payoff function is kept in its general form for the first part of the analysis in order to demonstrate the generality of the solution. A note on the total number of end conditions that can be specified is in order at this point. First, recall that there are a total of five differential equations in the set f_i^*. Hence, a total number of $2n + 2 = 12$ (where $n = 5$) boundary conditions must be considered. Naturally all 12 conditions cannot be specified for there would be no optimization problem. Thus, it can be stated that at most eleven end conditions can be specified, leaving at least one degree of freedom for the optimization.

As before, the first step in the solution of this problem is the construction of the augmented function

$$F = \sum_{i=1}^{6} p_i f_i^* \tag{2.82}$$

or

$$F = p_1(\dot{x} - w) + p_2(\dot{h} - v) + p_3\left[\dot{w} - \frac{cu}{m}\cos\psi\right] + p_4\left(\dot{v} + g - \frac{cu}{m}\sin\psi\right) + p_5(\dot{m} + u) + p_6\left[u(u_{max} - u) - \dot{\alpha}\right] \tag{2.83}$$

Now in a manner completely analogous to that presented in the discussion of the previous vertical shoot problem, the Euler-Lagrange equation is applied for each dependent variable. Since there are eight dependent variables, this application results in eight equations, five of which for this case are differential equations. The results are

$$\dot{p}_1 = 0 \tag{2.84}$$

$$\dot{p}_2 = 0 \tag{2.85}$$

$$\dot{p}_3 = -p_1 \tag{2.86}$$

$$\dot{p}_4 = -p_2 \tag{2.87}$$

$$\dot{p}_5 = \left(\frac{cu}{m^2}\right) K_\psi \tag{2.88}$$

$$0 = -K_\beta + p_6 \left(u_{max} - 2u\right) \tag{2.89}$$

$$0 = p_6 \alpha \tag{2.90}$$

$$0 = p_3 \sin\psi - p_4 \cos\psi \tag{2.91}$$

where

$$K_\beta \equiv \frac{c}{m}\left(p_3 \cos\psi + p_4 \sin\psi\right) - p_5 \tag{2.92}$$

and

$$K_\psi \equiv p_3 \cos\psi + p_4 \sin\psi \tag{2.93}$$

Now, as in the vertical shoot problem, the first integral is easily obtained because the augmented function is formally independent of time. This integral is:

$$p_1 w + p_2 v - p_4 g + K_\beta u = C \tag{2.94}$$

The transversality condition can also be applied in the same manner employed in the previous problem. The result is:

$$\left[d\phi - \phi\, dt \;\; p_1\, dx + p_2\, dh + p_3\, dw + p_4\, dv + p_5\, dm\right]_0^f = 0 \tag{2.95}$$

The term $d\phi$ is undefined so far, since no performance function has been defined. If some function were to be specified at this point, the term $d\phi$ would merely introduce additional terms in Equation (2.95). The most general form of these terms would be

$$d\phi = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial h}dh + \frac{\partial\phi}{\partial w}dw + \frac{\partial\phi}{\partial v}dv + \frac{\partial\phi}{\partial m}dm \tag{2.96}$$

where for generality ϕ is assumed to be a function of t, X, h, w , v , and m. As in the vertical shoot problem, the variables that have specified end conditions have zero differentials in the transversality equation, e.g., if m is specified, then dm = 0. The terms that contain differentials of variables whose end conditions are not specified have to have coefficients equal to zero, since the differential is arbitrary, e.g., if the term is C dt, and t is not specified at the end condition, then C must be zero. A specific application of a typical ϕ will be made later on.

It is now desired to obtain an expression for $\dot{K}_\beta$ that applies to the entire extremal arc. This expression is obtained by finding the time derivative of Equation (2.92) and utilizing the results of the Euler-Lagrange equations [(2.86), (2.87), (2.88), (2.91)] and Equation (2.93). The details are sketch below:

First,

$$\dot{K}_\beta = -\frac{c(p_3 \cos\psi - p_4 \sin\psi)}{m^2}\dot{m} + \frac{c}{m}\cos\psi\,\dot{p}_3 + \frac{c}{m}\sin\psi\,\dot{p}_4 + \Big[\underbrace{\Big|\frac{c}{m}p_3 \sin\psi\Big| + \frac{c}{m}p_4 \cos\psi}_{=0}\Big]\dot{\psi} - \dot{p}_5 \tag{2.97}$$

But, recalling that

$$\begin{aligned} \dot{m} &= -u, \\ \dot{p}_3 &= -p_1, \\ \dot{p}_4 &= -p_2, \end{aligned}$$

and

$$\dot{p}_5 = \frac{cu}{m^2}(p_3 \cos\psi + p_4 \sin\psi)$$

the following relation results

$$\dot{K}_\beta = \frac{c}{m}(p_1 \cos\psi + p_2 \sin\psi) \tag{2.98}$$

Now as in the previous problem, it can be shown that there are no extremal subarcs for which a variable thrust is the optimum policy. This conclusion is a direct result of Equation (2.90) which states that either p_6 or α must be zero at any one given time. If $\alpha = 0$ then U must be either full on or full off, i.e.,

$$u = 0 \quad \text{OR} \quad u = u_{max} \tag{2.99}$$

The other possibility is $p_6 = 0$, which leads to

$$K_\beta = 0 \quad \dot{K}_\beta = 0 \tag{2.100}$$

This possibility is not consistent with Equation (2.98) so there cannot be extremal subarcs for which $p_6 = 0$, or consequently there are no subarcs flown with a variable thrust.

Now that the magnitude of the thrust has been determined to be of the "full on" - "full off" type, it is interesting to determine some information about the direction of the thrust. An inspection of the first four Euler-Lagrange Equations [(2.84), (2.85), (2.86), (2.87)] yields the following expressions for p_1, p_2, p_3, and p_4 by straightforward integration:

$$p_1 = C_1 \tag{2.101}$$

$$p_2 = C_2 \tag{2.102}$$

$$p_3 = C_3 - C_1 t \tag{2.103}$$

$$p_4 = C_4 - C_2 t \tag{2.104}$$

where C_1, C_2, C_3, and C_4 are constants of integration. The solution for the steering angle is obtained from the last Euler-Lagrange Equation, (1.70). It is

$$p_3 \sin \psi = p_4 \cos \psi \tag{2.105}$$

or

$$\tan \psi = \frac{p_4}{p_3}$$

From Equations (2.103) and (2.104), it is seen that the tangent of the thrust inclination with respect to the horizon is a bilinear function of time, i.e.,

$$\tan \psi = \frac{C_4 - C_2 t}{C_3 - C_1 t} \tag{2.106}$$

It is recalled that all of the results obtained so far have been derived without reference to a specific payoff function, ϕ. Hence, the bilinear tangent steering law is a general result. Several particular cases can be considered in order to show the type of specific steering law that results under more specific payoff conditions.

Case 1 Payoff Independent of Velocity

Assume that the particular payoff function chosen does not depend on w or v, the horizontal and vertical velocities, respectively. Then the differential terms for w and v in Equation (1.75) are

$$\frac{\partial \phi}{\partial w} = 0 \qquad \frac{\partial \phi}{\partial v} = 0$$

Assuming the most general form for ϕ, the transversality condition, therefore,

$$\left[\frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial h} dh + \frac{\partial \phi}{\partial m} dm - C dt + p_1 dx + p_2 dh + p_3 dw \; p_4 dv + p_5 dm\right]_o^f = 0 \tag{2.107}$$

But, if the terminal velocity is not specified, dw and dv will not be zero; hence,

$$p_3^f = p_4^f = 0$$

Referring to Equations (2.103) and (2.104), this consequence implies that the following equations must be true at the terminal point:

$$C_3 = C_1 t^f$$
$$C_4 = C_2 t^f$$

Now, the steering equation becomes

$$\tan \psi = \frac{C_2 t^f - C_2 t}{C_1 t^f - C_1 t} = \frac{C_2 (t^f - t)}{C_1 (t^f - t)} = \frac{C_2}{C_1} \tag{2.108}$$
$$= CONSTANT$$

This result shows that as long as the payoff function is independent of the velocity of the vehicle and the final velocity of the vehicle is not specified, then the optimum thrust direction with respect to the horizon is constant for the entire trajectory. Of course, the constant will vary depending on the mission and the nature of the payoff function. However, it is interesting to note that whatever payoff function is chosen (as long as it is independent of velocity), the same type of policy is optimum.

Case 2 Payoff Independent of Velocity and Time

As in the previous case, consider the case where the payoff function is independent of velocity but in addition is also independent of time. The following conditions must exist for the optimal solution

$$\frac{\partial \phi}{\partial w} = 0$$

$$\frac{\partial \phi}{\partial v} = 0$$

$$\frac{\partial \phi}{\partial t} = 0$$

w^f, v^f, t^f unspecified

Now, as in the previous case, the transversality condition becomes

$$\left. \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial h} dh + \frac{\partial \phi}{\partial m} dm + c\,dt + p_1 dx + p_2 dh + p_3 dw + p_4 dv + p_5 dm \right]_0^f = 0 \tag{2.109}$$

But, since the final velocities and time are not specified Equation (2.109) can be satisfied if

$$p_3^f = p_4^f = c = 0 \tag{2.110}$$

At this point, it is assumed that the final point of the trajectory is reached by coasting; thus, the first integral Equation (2.94) becomes

$$p_1^f w^f + p_2^f v^f = 0 \tag{2.111}$$

because

$p_{4_f} = 0$ from Equation (2.110)

$u_f = 0$ from assumption

and $c_1 = 0$ from Equation (2.110)

But since the differential equations for p_1 and p_2 [Equations (2.84) and (2.85)] imply that p_1 and p_2 are constant, Equation (2.11) becomes

$$p_1 w^f + p_2 v^f = 0 \tag{2.112}$$

or

$$\frac{p_1}{p_2} = -\frac{v^f}{w^f} \tag{2.113}$$

Equations (2.101) through (2.104) gave the general forms of the equations for p_1, p_2, p_3, and p_4. The boundary conditions of Equation (2.110) determine two of the constants:

$$C_3 = p_1 t^f \tag{2.114}$$

$$C_4 = p_2 t^f \tag{2.115}$$

Hence,

$$p_3 = p_1 t^f - p_1 t \tag{2.116}$$

$$p_4 = p_2 t^f - p_2 t \tag{2.117}$$

where p_1 has been used for C_1 and p_2 for C_2. Finally, the steering policy for this case can be obtained as

$$\tan \psi = \frac{p_1 t^f - p_1 t}{p_2 t^f - p_2 t} = \frac{p_1}{p_2} \tag{2.118}$$

Using Equation (2.113),

$$\tan \psi = -\frac{v^f}{w^f}$$

Therefore, it can be stated that for the case when the payoff function is not a function of the velocity or time and if the terminal values of these same parameters are not specified, then the optimal steering policy is to thrust in a constant direction which is perpendicular to the final velocity vector. This result poses a problem since it was assumed for this case that the final velocity was unspecified; thus, it is necessary to construct the solution by the trial and error. (It is known that the thrust angle is constant throughout the trajectory but the value of the angle is not known. So, various values of the angle must be tried until one is found such that the final velocity vector is perpendicular to the thrust chosen.)

Case 3 Payoff Independent of Velocity but the Initial Velocity Magnitude is Specified

Now consider the case where the payoff function ϕ is independent of velocity, as before, i.e.,

$$\frac{\partial \phi}{\partial w} = 0 \qquad \frac{\partial \phi}{\partial v} = 0 \tag{2.119}$$

Once again, the transversality condition becomes

$$\left. \begin{aligned} &\frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial h} dh + \frac{\partial \phi}{\partial m} dm - C dt \\ &+ p_1 dx + p_2 dh + p_3 dw + p_4 dv + p_5 dm \end{aligned} \right]_0^f = 0 \tag{2.120}$$

or

$$p_3^f dw^f + p_4^f dv^f - p_3^0 dw^0 - p_4^0 dv^0 = 0 \tag{2.121}$$

This information must now be combined with the initial constraint on the velocity to construct the desired solution, i.e.,

$$(w^0)^2 + (v^0)^2 = \text{CONSTANT} \tag{2.122}$$

or

$$2 w^0 dw^0 + 2 v^0 dv^0 = 0 \tag{2.123}$$

Equations (2.121) and (2.123) are a simultaneous set of equations

$$\left. \begin{array}{l} p_3^0 dw^0 + p_4^0 dv^0 = 0 \\ w^0 dw^0 + v^0 dv^0 = 0 \end{array} \right\} \tag{2.124}$$

whose solution is

$$p_3^0 v^0 - p_4^0 w^0 = 0 \tag{2.125}$$

This relation leads to the result

$$\frac{v^\circ}{w^\circ} = \frac{p_4^\circ}{p_3^\circ} = \tan \psi^\circ \tag{2.126}$$

This equation states that the optimum initial velocity is parallel to the thrust for this case.

Case 4 Payoff Independent of Horizontal Coordinate of Terminal Position

Consider now the case in which the payoff function is not a function of the horizontal component of the terminal position. That is,

$$\frac{\partial \phi}{\partial X} = 0 \tag{2.127}$$

Again referring to the transversality condition, it is seen that for this case

$$p_1^f = 0$$

This condition follows from the fact that dX need not be zero, since X was not specified. Since p_1 was determined to be constant (equal to C_1), it is seen that $C_1 = 0$. The expression for the thrust direction then becomes

$$\tan \psi = \frac{C_4 - C_2 t}{C_3} = \frac{C_4}{C_3} - \frac{C_2}{C_3} t \tag{2.128}$$

This result states that if the payoff function is independent of the horizontal coordinate, the tangent of the thrust inclination angle is a linear function of time.

Case 5 Payoff Independent of Altitude

The last case to be considered is the one in which the payoff function is independent of the altitude, h, and the final altitude is not specified. The transversality condition leads to

$$p_2^f = 0 \tag{2.129}$$

But, p_2 is constant throughout the problem and is equal to C_2. Hence, the general expression for the steering angle becomes

$$\tan \psi = \frac{C_4}{C_3 - C_1 t} \tag{2.130}$$

or

$$\cot \psi = \left(\frac{C_3}{C_4}\right) - \left(\frac{C_1}{C_4}\right) t \tag{2.131}$$

This states that for the case in which the pay-off function is independent of altitude, the optimum steering policy is one whose cotangent of the thrust angle is a linear function of time.

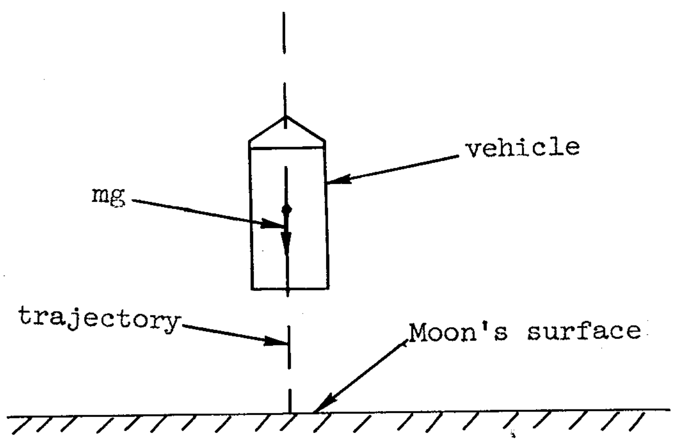
Although the assumptions that were made in the previous problem helped to simplify the problem considerably, the analyses still presented the basic concepts involved in the solution of the optimization problem and a somewhat closed-form solution; that is, although a unique solution was not attainable, the form of the solution was known to within some constants that are determined by the specific mission. In many cases, the only way that the unique solution can be found for the particular problem is by trial and error. However, this solution process is not difficult when the form of the solution is known.

2.2.4 Optimal Thrust Programming for Soft Landing

As another example, consider the problem of soft landing a vehicle on a planet with the minimal amount of fuel. The vehicle is assumed to be throttleable, and again the thrust must lie in the range

$$0 \leq T \leq T_{max}$$

The following sketch shows the vehicle during a soft landing.



The following assumptions are made:

(1) The only forces acting on the vehicle are those of gravity and the thrust.

(2) The thrust is tangent to the descent trajectory, which is vertical.

(3) The surface is flat near the landing point.

(4) Gravity is constant during the landing.

(5) Exhaust velocity is constant

(6) The mass flow rate of the propellant can be varied between zero and some fixed upper limit.

The thrust can be expressed as

$$T = T_{max}\, u = -c\,\dot{m} \tag{2.132}$$

where T = thrust

T_{max} = maximum value of the thrust

u = throttle setting

c = exhaust velocity

$\dot{m}$ = mass flow rate

The equations of motion for the vehicle can be written as

$$\ddot{x} = \frac{T_{max}\, u}{m} - g \tag{2.133}$$

$$\dot{m} = \frac{-T_{max}\, u}{c} \tag{2.134}$$

These equations can be written in state variable form by introducing the variables

X_1 = altitude

X_2 = velocity

X_3 = mass

The state equations then become

$$\dot{X}_1 = x_2 \tag{2.135}$$

$$\dot{X}_2 = \frac{T_{max}\, u}{X_3} - g \tag{2.136}$$

$$\dot{X}_3 = -T_{max}\, u/c \tag{2.137}$$

The previous examples in this section have employed a Calculus of Variation formulation; however, this problem will be formulated by using the hamiltonian and the Pontryagin Maximum Principle. Though these methods are equivalent this approach sometimes provides more insight to the choice of control to be made.

The hamiltonian for the system is

$$H = \sum p_i f_i = p_1 x_2 + p_2 \left[\frac{T_{max} u}{x_3} - g \right] - p_3 \frac{T_{max} u}{c} \qquad (2.138)$$

and the adjoint equations are found to be

$$\dot{p}_i = - \frac{\partial H}{\partial x_i} \qquad (2.139)$$

or

$$\dot{p}_1 = 0 \qquad (2.140)$$

$$\dot{p}_2 = -p_1 \qquad (2.141)$$

$$\dot{p}_3 = \frac{p_2 T_{max} u}{x_3^2} \qquad (2.142)$$

Now, since in a soft landing the terminal velocity must be zero when the altitude is zero, the terminal conditions are

$$\left.\begin{aligned} X_1 &= 0 \\ X_2 &= 0 \\ X_3 &= \text{maximum} \end{aligned}\right\} \quad \text{at } t = t^f \qquad (2.143)$$

But, any arbitrary initial conditions can be imposed as long as the assumptions are not violated. Hence, the initial conditions are

$$\left.\begin{aligned} X_1 &= X_1^o \\ X_2 &= X_2^o \\ X_3 &= X_3^o \end{aligned}\right\} \quad \text{at } t = t^o \qquad (2.144)$$

It is noted that a specific time has not been stipulated. Thus, the time and mass provide two degrees of freedom; the time gives a degree of freedom in the end conditions and the mass gives a degree of freedom in the state variables. The payoff function is quite simple for this problem since it is a function of the final mass, i.e.,

$$J = -X_3^f \qquad (2.145)$$

Minimizing the negative of the final mass is equivalent to maximizing the final mass which in turn is equivalent to minimizing the propellant consumed.

Rewriting the hamiltonian as

$$H = T_{max}\, u \left[\frac{p_2}{x_3} - \frac{p_3}{c}\right] + p_1 x_2 - p_2 g \quad , \tag{2.146}$$

it is seen that the hamiltonian is maximized for each point in time if U is chosen to be zero when the term in brackets is negative and U chosen as "one" if the term in the brackets is positive. Stated mathematically,

$$U = \begin{cases} 0 & \text{for } K < 0 \\ 1 & \text{for } K > 0 \\ \text{arbitrary} & \text{for } K = 0 \end{cases} \tag{2.147}$$

where $K = \frac{p_2}{x_3} - \frac{p_3}{c}$

Thus, it is seen that K is a switching function which can be used to determine whether the thrust is to be on or off. Further, it can be shown (reference 1.2) that K cannot be zero for any finite closed time interval. That is, when the switching function, K , passes through zero, it does so instantaneously. Hence, there is no time period for which an intermediate throttle setting is permissible and as a result, the thrust is either "full on" or "full off". This type of control is referred to as "bang-bang" control. Reference 1.2 also shows that depending on the initial conditions, there is at most one switching point in the problem and the control is either "full on" from the initiation of the mission until touchdown or there is a period of zero thrust followed by full thrust until touchdown.

Before discussing the implementation of the control, the results to this point, will be summarized. A set of six differential equations, three of which are the differential constraint (state) equations and three of which are the adjoint differential equations has been formulated. This set is:

$$\dot{X}_1 = X_2 \tag{2.148}$$

$$\dot{X}_2 = (T_{MAX}\, u / x_3) - g \tag{2.149}$$

$$\dot{X}_3 = -T_{MAX}\, u / c \tag{2.150}$$

$$\dot{P}_1 = 0 \tag{2.151}$$

$$\dot{P}_2 = -P_1 \tag{2.152}$$

$$\dot{P}_3 = p_2\, T_{MAX}\, u / x_3^2 \tag{2.153}$$

and the boundary conditions mentioned are

$$\left.\begin{aligned} X_1 &= X_1^o \\ X_2 &= X_2^o \\ X_3 &= X_3^o \end{aligned}\right\} \quad \text{at } t = t^o \tag{2.154}$$

$$\left.\begin{aligned} X_1 &= 0 \\ X_2 &= 0 \end{aligned}\right\} \quad \text{at } t = t^f \tag{2.155}$$

Some additional boundary condition information may be obtained from the adjoint equations at $t = t^f$ from the transversality condition

$$p_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0 \tag{2.156}$$

where ψ_j and ϕ are

$$\psi_1 = X_1^f - 0 = 0 \tag{2.157}$$

$$\psi_2 = X_2^f - 0 = 0 \tag{2.158}$$

$$\phi = -X_3^f \tag{2.159}$$

So equation (1.135) yields

$$\left.\begin{aligned} p_1 + \mu_1 &= 0 \\ p_2 + \mu_2 &= 0 \\ p_3 - 1 &= 0 \end{aligned}\right\} \quad \text{at } t = t^f \tag{2.160}$$

However, since μ_1 and μ_2 are unspecified multipliers, the first two conditions give no additional information, but the last yields another **boundary condition at $t = t^f$, $p_3 = 1$. Hence, there are a total of six boundary conditions**

$$\left.\begin{aligned} X_1 &= X_1^o \\ X_2 &= X_2^o \\ X_3 &= X_3^o \end{aligned}\right\} \quad \text{at } t = t^o \tag{2.161}$$

$$\left.\begin{aligned} X_1 &= 0 \\ X_2 &= 0 \\ P_3 &= 1 \end{aligned}\right\} \quad \text{at } t = t^f \tag{2.162}$$

Further, it will be shown in Equation (2.214) - (2.219) that the hamiltonian must be zero over the entire trajectory since $\dot{H} = 0$ and

$$H = \mu_j \frac{\partial \psi_j}{\partial t} + \frac{\partial \phi}{\partial t} = 0 \text{ at } t = t^f \tag{2.163}$$

Normally, at this point in a trajectory problem, the remainder of the solution must be handled numerically (i.e., be solved by numerical techniques such as the gradient method or quasilinearization). However, in the problem at hand, it is fortunate that a graphical solution is possible. For completeness, the following discussion will outline the graphical solution and its mechanization on a vehicle.

First, consider the velocity history of the vehicle that starts from some altitude $X_1{}^*$ with a velocity $X_2{}^*$ and mass M_o at the initiation of a constant thrust; i.e., the mass decreases linearly with time

$$m = x_3(t) = M_o - \dot{m}t$$

Now, recalling that $\frac{\dot{m}}{m} = \frac{d}{dt}(\ln m)$ equation (2.133) can be written as

$$\ddot{x} = \frac{T_{max}\, u}{m} - g = -\frac{\dot{m}c}{m} - g = -c\frac{d}{dt}(\ln m) \tag{2.164}$$

Thus, integration yields

$$\dot{x}(t) = c \ln \frac{m(t)}{m(0)} - gt + \dot{x}(0) \tag{2.165}$$

and

$$x_2(t) = -c \ln\left[\frac{M_o - \dot{m}t}{M_o}\right] - gt + x_2^* \tag{2.166}$$

$$= -c \ln\left[1 - \frac{\dot{m}}{M_o}t\right] - gt + x_2^*$$

But, the first state equation yields

$$x_1(t) = \int_0^t x_2(\xi)\, d\xi + x_1^*$$
$$= \frac{cM_0}{\dot{m}} \left(1 - \frac{\dot{m}}{M_0} t\right) \ln \left(1 - \frac{\dot{m}}{M_0} t\right) + ct$$
$$- \frac{1}{2} gt + x_2^* t + x_1^* \tag{2.167}$$

Now, these two solutions must be matched at landing, i.e.,

$$x_1(t_1) = x_2(t_1) = 0 \tag{2.168}$$

Thus, equations (2.166) and (2.167) become

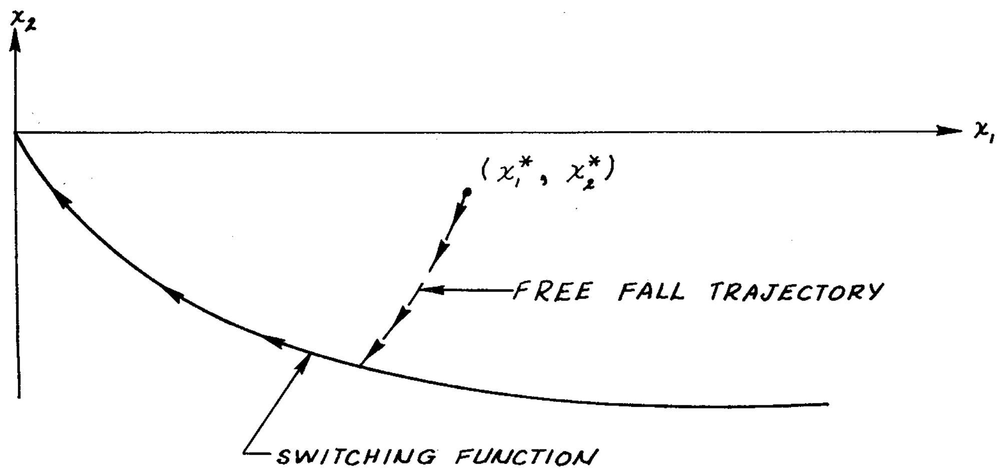
$$0 = -c \ln \left(1 - \frac{\dot{m}}{M_0} t_1\right) - g t_1 + x_2^* \tag{2.169}$$
$$0 = \frac{cM_0}{\dot{m}} \left(1 - \frac{\dot{m}}{M_0} \cdot t_1\right) \ln \left(1 - \frac{\dot{m}}{M_0} t_1\right) + ct_1$$
$$- \frac{1}{2} g t_1^2 + x_2^* t_1 + x_1^* \tag{2.170}$$

These equations yield expressions for X_1^* and X_2^* as

$$x_1^* = - \frac{c M_0}{\dot{m}} \ln \left(1 - \frac{\dot{m}}{M_0} t_1\right) - ct_1 - \frac{1}{2} g t^2 \tag{2.171}$$

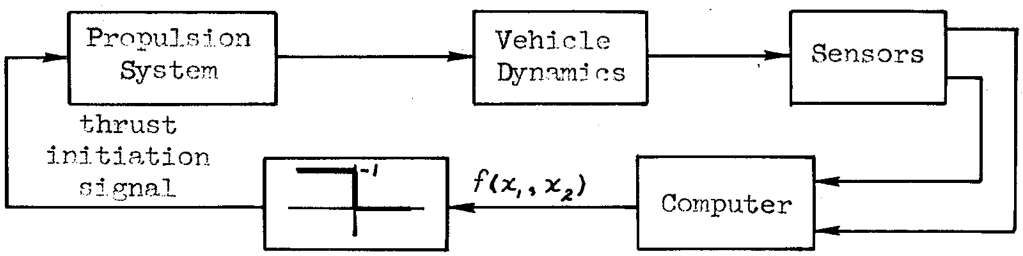
$$x_2^* = c \ln \left(1 - \frac{\dot{m}}{M_0} t_1\right) + gt_1 \tag{2.172}$$

By eliminating t_1 from these equations, the switching function can be obtained. It is sketched below (reference 1.2).



If the landing vehicle starts at some state above the switching curve, (X_1^*, X_2^*), the optimal thrusting policy is to allow free fall along the dotted trajectory until the state of the vehicle intersects the switching curve. The thrust should then be applied "full on" for the remainder of the descent. If the initial state of the vehicle is on the switching curve, the thrust should be applied immediately for a soft landing. If the initial state is below the switching curve, the vehicle cannot make a soft landing with the available thrust.

It is seen that the knowledge of the present state and its relationship to the switching curve is all that is necessary to accomplish the soft landing. This knowledge can be attained by employing sensors to measure altitude and altitude rate and can be mechanized as shown in the block diagram below.

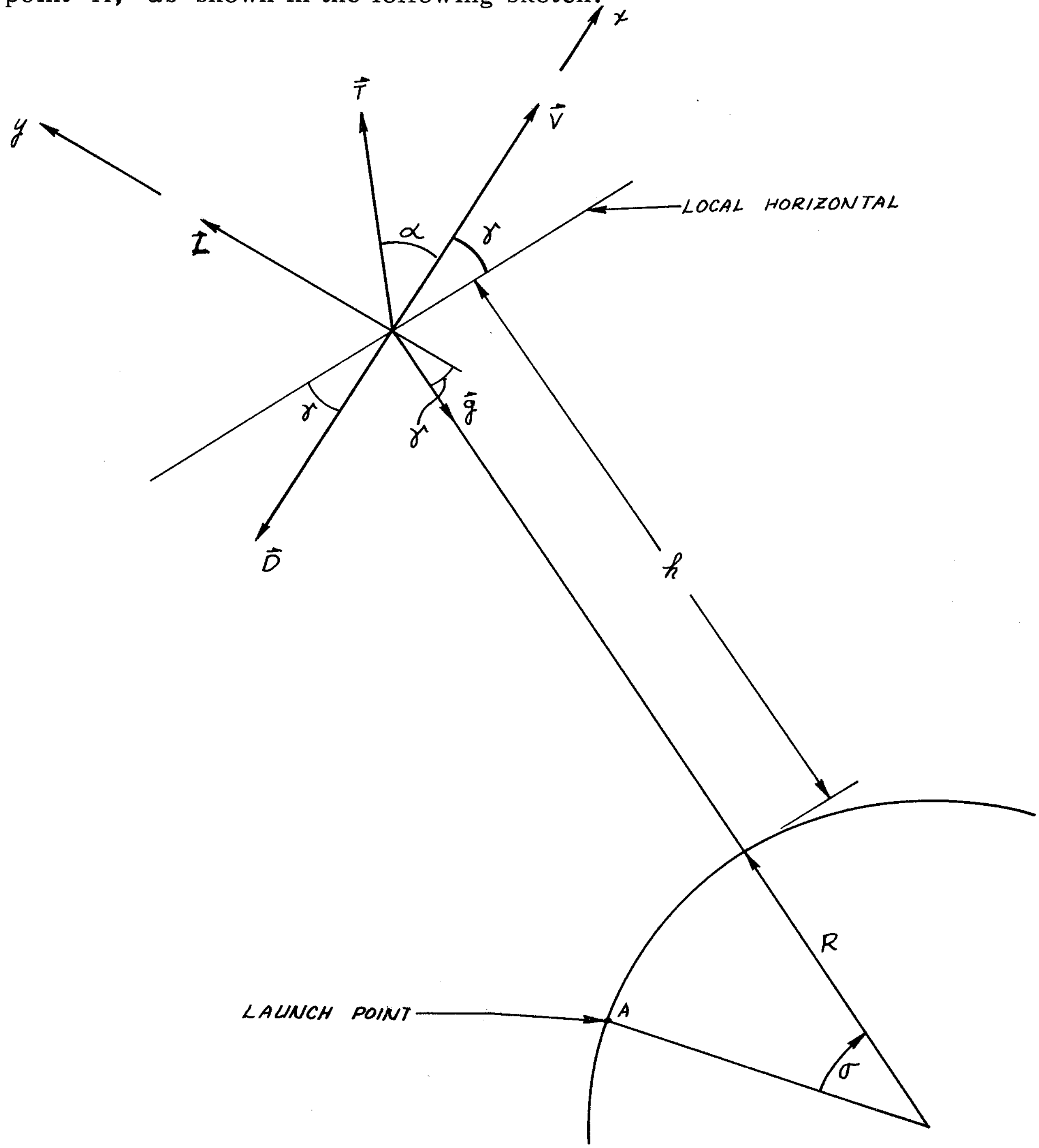


To summarize the problem, it is seen that optimum control is of the "bang-bang" type with one switching point. With this information, the switching curve was then found by writing the equations of motion for the descent of a vehicle with the rocket "full on" and specifying the terminal position and velocity as zero. By eliminating the parameter, t, a functional relationship for the switching curve could be obtained in terms of x_1, and x_2.

Finally, this function was used as a switching criterion for initial states that are above the switching curve.

2.2.5 Nonlinear Boost Problem

In order to demonstrate the solution to a more general nonlinear trajectory optimization problem, a boost mission will now be analyzed. Some of the restrictions of the previous section will be relaxed; however, the motion will be confined to a plane and a non-rotating atmosphere; and a central force field will be assumed. Consider a vehicle that is launched from some point A, as shown in the following sketch.



Let σ be the angle between the launch point and the current position and let γ be the angle between the local horizontal and the velocity vector (flight path angle). Thus, if a coordinate system is constructed at the vehicle's center of mass with the X axis aligned with the velocity vector, the rotation of the coordinate system will be

$$\omega = \dot{\gamma} - \dot{\sigma} \tag{2.173}$$

The equations of motion for the vehicle then become

$$\frac{\vec{F}}{m} = \frac{d}{dt}(\vec{V}) = \frac{\partial \vec{V}}{\partial t} + \vec{\omega} \times \vec{V} \tag{2.174}$$

where $\vec{F}$ = total force on vehicle
m = mass of vehicle
$\vec{V}$ = inertial velocity of vehicle

$\frac{\partial (\cdot)}{\partial t} = (\dot{\cdot})_x \vec{i} + (\dot{\cdot})_Y \vec{j} + (\dot{\cdot})_z \vec{k}$ in vehicle coordinate system.

Now, since $\bar{\omega} = (\dot{\gamma} - \dot{\sigma})\bar{k}$, Equation (2.174) in component form becomes

$$\begin{bmatrix} \frac{F_X}{m} \\ \\ \frac{F_Y}{m} \end{bmatrix} = \begin{bmatrix} \dot{V} \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ (\dot{\gamma} - \dot{\sigma})V \end{bmatrix} \tag{2.175}$$

where the forces acting on the vehicle are thrust, drag, lift, and gravity. The thrust will be designated as before

$$T = T_{MAX}\, u \tag{2.176}$$

and, the lift and drag are known functions of V, h, and α of the form (Reference 1.6)

$$L = \frac{1}{2} \rho V^2 S \alpha\, C_{L\alpha}\left(\frac{V}{a}\right) \tag{2.177}$$

$$D = \frac{1}{2} \rho V^2 S \left\{ C_{D_\theta} + \alpha^2 C_{L\alpha} \right\} \qquad (2.178)$$

where ρ = p (h) = air density

a = a (h) = speed of sound

S = reference area of vehicle

$C_{D\theta} = C_{DO}\left(\frac{V}{a}\right)$ drag coefficient

$C_{L\alpha} = C_{L\alpha}\left(\frac{V}{a}\right)$ lift coefficient

Finally, the gravity on the vehicle is given by the relation

$$g = g_o \frac{R^2}{(R+h)^2} \qquad (2.179)$$

where g_o is the gravity at the radius R.

Now that the forces on the vehicle have been defined, the summation of forces can be written and combined with the equations of motion. This process applied to the X coordinate yields

$$\dot{V} = \frac{T \cos \alpha}{m} - \frac{D}{m} - g_o \frac{R^2}{(R+h)^2} \sin \gamma \qquad (2.180)$$

Similarly, in the Y direction

$$(\dot{\gamma} - \dot{\sigma}) V = \frac{T \sin \alpha}{m} + \frac{L}{m} - g_o \frac{R^2}{(R+h)^2} \cos \gamma \qquad (2.181)$$

But

$$\dot{h} = V \sin \gamma \qquad (2.182)$$

$$\dot{\sigma} = \frac{V \cos \gamma}{R+h} \qquad (2.183)$$

$$\left.\begin{aligned} V(0) &= 0 \\ \gamma(0) &= \text{unspecified} \\ h(0) &= 0 \\ \sigma(0) &= \sigma^o \\ m(0) &= m^o \end{aligned}\right\} \quad \text{at } t = 0 \tag{2.192}$$

Then, the terminal conditions are defined to be:

$$\left.\begin{aligned} V(t^f) &= V^f \\ \gamma(t^f) &= \text{unspecified} \\ h(t^f) &= h^f \\ \sigma(t^f) &= \sigma^f \\ m(t^f) &= \text{Unspecified} \end{aligned}\right\} \quad t^f \text{ unspecified} \tag{2.193}$$

The optimization problem is now to minimize the fuel used to satisfy the terminal conditions; i. e., the state variable m is to be maximized at the terminal point. Conversely, since the payoff function is to be minimized, the negative of the mass is chosen to be the payoff function.

$$J = -m_f \tag{2.194}$$

This problem will be solved using the Pontryagin Maximum Principle. Thus, the first step is the construction of the hamiltonian.

$$H = p_V \dot{V} + p_\gamma \dot{\gamma} + p_h \dot{h} + p_\sigma \dot{\sigma} + p_m \dot{m} \tag{2.195}$$

The co-state equations can then be determined from the hamiltonian as

$$\dot{p}_V = -\frac{\partial H}{\partial V} = p_\gamma \left\{ \left[\frac{T_{max}\, u \sin\alpha}{m} + \frac{L}{m} - g_o \frac{R^2 \cos\gamma}{(R+h)^2} \right] \left(\frac{-1}{V^2} \right) + \frac{\cos\gamma}{(R+h)} \right\} \tag{2.196}$$
$$- p_h \sin\gamma - p_\sigma \frac{\cos\gamma}{(R+h)}$$

$$\dot{p}_\gamma = -\frac{\partial H}{\partial \gamma} = -p_V \left[-g_o \frac{R^2}{(R+h)^2} \cos\gamma \right] - p_\gamma \left\{ \left[g_o \frac{R^2 \sin\gamma}{(R+h)^2 V} \right] \right. \tag{2.197}$$
$$\left. - \frac{V \sin\gamma}{(R+h)} \right\} - p_h V \cos\gamma - p_\sigma \left[\frac{V \sin\gamma}{(R+h)} \right]$$

Thus, Equation (2.183) permits Equation (2.181) to be written as

$$\dot{\gamma} = \frac{T \sin \alpha}{m V} + \frac{L}{m V} - g_0 \frac{R^2 \cos \gamma}{(R+h)^2 V} + \frac{V \cos \gamma}{R+h} \qquad (2.184)$$

Now noting that the thrust is

$$T = T_{MAX} \, U = - c \dot{m} \qquad (2.185)$$

allows the mass flow rate to be written as

$$\dot{m} = \frac{- T_{MAX} \, U}{C} \qquad (2.186)$$

The differential constraint equations are now known. For convenience they will be repeated together.

$$\dot{V} = \frac{T_{max} \, U \cos \alpha}{m} - \frac{D}{m} - g_0 \frac{R^2}{(R+h)^2} \sin \gamma \qquad (2.187)$$

$$\dot{\gamma} = \frac{T_{max} \, U \sin \alpha}{m V} + \frac{L}{m V} - g_0 \frac{R^2 \cos \gamma}{(R+h)^2 V} + \frac{V \cos \gamma}{(R+h)} \qquad (2.188)$$

$$\dot{h} = V \sin \gamma \qquad (2.189)$$

$$\dot{\sigma} = \frac{V \cos \gamma}{R+h} \qquad (2.190)$$

$$\dot{m} = \frac{T_{max} \, U}{c} \qquad (2.191)$$

It should be noted that some of the coefficients involved in these equations must be either calculated from an approximation or located in a table as the computation progresses.

The next items of interest in this problem are the boundary conditions and the payoff function. First, the initial state vector is defined to be

$$\dot{p}_h = -\frac{\partial H}{\partial h} = -p_V\left(\frac{2 g_0 R^2 \sin\gamma}{(R+h)^3}\right) - p_\gamma\left\{\left(\frac{2 g_0 R^2 \cos\gamma}{V(R+h)^3}\right) - \frac{V\cos\gamma}{(R+h)^2}\right\} \tag{2.198}$$
$$- p_\sigma\left(\frac{-V\cos\gamma}{(R+h)^2}\right)$$

$$\dot{p}_\sigma = -\frac{\partial H}{\partial \sigma} = 0 \tag{2.199}$$

$$\dot{p}_m = -\frac{\partial H}{\partial m} = -p_V\left[\frac{-T_{max}\, U \cos\alpha}{m^2} + \frac{D}{m^2}\right] \tag{2.200}$$
$$- p_\gamma\left[\frac{-T_{max}\, U \sin\alpha}{V m^2} \quad \frac{L}{m^2 V}\right]$$

Note that there are two control variables in the problem, α and U, and that it is desired to maximize the hamiltonian for all instants in time $t^o \le t \le t^f$ subject to the control constraint. If there were no control constraints, the control that maximizes the hamiltonian could be found by setting each of the partial derivatives of H with respect to α and U to zero. This procedure would give the values of α and U in terms of other system variables. However, if there are control constraints, the constraint equations must be joined to the hamiltonian before the derivatives are taken. In this case, the control constraint is relatively simple and the choice of the control could be made without joining the control constraint. However, a formal approach will be taken and the constraint presented earlier will be joined for demonstration purposes so that the procedure for more complex control constraints can be seen. Thus, the augmented hamiltonian becomes

$$\bar{H} = H + \lambda\left(g(U) + \eta^2\right) \tag{2.201}$$

where η is an unknown variable used to satisfy the constraint equation. For the particular case of $0 \le U \le 1$, this hamiltonian becomes

$$\bar{H} = H + \lambda\left[U(1-U) + \eta^2\right] \tag{2.202}$$

If the hamiltonian of Equation (2.195) is used for H, the following partial derivatives result

$$\frac{\partial \bar{H}}{\partial U} = \frac{p_V T_{max} \cos \alpha}{m} + \frac{p_\gamma T_{max} \sin \alpha}{m V} + p_m \left(\frac{-T_{max}}{c}\right) + \lambda(1+2U) = 0 \qquad (2.203)$$

$$\frac{\partial \bar{H}}{\partial \alpha} = p_V \left(\frac{-T_{max} U \sin \alpha}{m V}\right) + p_\gamma \left(\frac{T_{max} U \cos \alpha}{m V}\right) = 0 \qquad (2.204)$$

$$\frac{\partial \bar{H}}{\partial \eta} = \lambda \, (2 \eta) = 0 \qquad (2.205)$$

Equations (2.202), (2.203), (2.204), and (2.205) provide four equations in the unknowns U, η, λ, and α. Thus, the control is determined for any instant providing the state and co-state variables for that instant are known.

At this point, a review of the procedure so far seems in order. First, the differential constraint equations were derived for this particular system. The adjoint differential equations were then found from the hamiltonian. Now, the two sets of differential equations can be integrated from the initial conditions over the entire trajectory. However, since all of boundary conditions are not specified, some will have to be guessed. As the integration proceeds, the values of the state and adjoint variables must be used in the set of simultaneous equations [equations (2.202), (2.203), (2.204), and (2.205)] to determine the optimum control for that particular point. This procedure continues until some set of conditions is satisfied. Obviously, the fact that some boundary conditions were guessed makes the chances of meeting all the terminal boundary conditions quite remote. However, linear theory can be used to correct the guessed boundary conditions and the equations can be integrated until the boundary conditions are satisfied and until the pay-off function ceases to improve from one iteration to the next. The detailed description of the numerical solution is extensively discussed in Reference (1.5). Since the nature of the numerical solution is beyond the scope of this monograph, it will not be pursued any further here.

Attention now turns to the boundary conditions that are imposed on the differential equations. The boundary conditions on the state variables are already known from the initial and final state variables specified in equations (2.192), (2.193), and (2.194). However, the boundary conditions on the co-state variables must be determined from

$$p_i + \mu_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0 \quad , \tag{2.206}$$

where the ψ_j's are known from Equation (2.193) to be

$$\left.\begin{aligned} \psi_v &= v - v^f = 0 \\ \psi_h &= h - h^f = 0 \\ \psi_\sigma &= \sigma - \sigma^f = 0 \end{aligned}\right\} \tag{2.207}$$

and where

$$\phi = -m_f \tag{2.208}$$

Thus, the resulting boundary conditions are

$$p_v = \mu_v \tag{2.209}$$

$$p_h = \mu_h \tag{2.210}$$

$$p_\sigma = \mu_\sigma \tag{2.211}$$

$$p_m = -1 \tag{2.212}$$

where the μ's are undetermined multipliers. The final value of the hamiltonian is found from

$$H = \mu_j \frac{\partial \psi_j}{\partial t} + \frac{\partial \phi}{\partial t} \tag{2.213}$$

But, since neither ψ_j nor ϕ are functions of time, the final value of the hamiltonian is zero. (If ψ_j and/or ϕ were functions of time, Equation (2.208) would supply additional boundary value information.)

Further, as was shown earlier, for this type of problem, $\dot{H}(t) = 0$. Now, consider the hamiltonian to be a function of $\underline{x}$, $\underline{p}$, $\underline{u}$, and t, i.e.,

$$H = H(t, \underline{x}, \underline{p}, \underline{u}) \tag{2.214}$$

So that time derivative of H is

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \underline{x}} \dot{\underline{x}} + \frac{\partial H}{\partial \underline{p}} \dot{\underline{p}} \frac{\partial H}{\partial \underline{u}} \dot{\underline{u}} \tag{2.215}$$

But, since the adjoint system is defined to be

$$\dot{\underline{p}}^T = -\frac{\partial H}{\partial \underline{x}} \tag{2.216}$$

and since

$$\dot{\underline{x}}^T = \frac{\partial H}{\partial \underline{p}} \tag{2.217}$$

the second and third terms cancel. Further, since the optimal control is selected such that

$$\frac{\partial H}{\partial \underline{u}} = 0 \tag{2.218}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{2.219}$$

Thus, if the hamiltonian is independent of time, the hamiltonian is constant over the entire optimal trajectory. This constant can be determined from the boundary conditions (Equation 2.213). However, for this case, H = 0.

The differential constraint equations and adjoint equations combined with the boundary conditions constitute a boundary value problem of the two point type with mixed end conditions. Thus, a solution to the nonlinear problem must be accomplished iteratively. The first step in the solution is to select a solution that satisfies either the boundary conditions or the constraint equations (depending on which numerical method is chosen).

Then, linear theory is used to modify the solution in a direction which tends to satisfy the other conditions. The theory of numerical techniques is extensively covered in Reference 1. 5.

2. 2. 6 Staging

The previous sections have considered the optimization of a single stage boost mission. However, since most boost missions are of the multistage type, it is desirable to extend the previous theory to missions of more than one stage. The staging problem poses the question of how big should each stage of a multistage booster be; that is, how much propellant should be placed in each stage. Equivalent questions would be how long should each stage burn and what size should each stage be.

Since the empty stage is dropped before a subsequent stage is used, there is a mass discontinuity in m (t), the mass state variable. However, the standard variational approach can be used for the optimization of such a problem if the discontinuity is interpreted properly and of the corner conditions (Weierstrass - Erdmann Corner conditions) are satisfied. The approach involves modification of the cost function. The payoff functions for a two stage mission can first be written in two parts; one from lift-off, t^o , to the staging point, t^s , and the other from the staging point to the end of the mission, t^f

$$J = \phi + \mu_f \psi_f + \int_{t^o}^{t^{s-}} F_1(x, \dot{x}, t)\, dt + \int_{t^{s+}}^{t^f} F_2(x, \dot{x}, t)\, dt \tag{2. 220}$$

Thus, the first variation in J around the staging point, δJ , becomes:

$$\delta J = \int_{t^o}^{t^{s-}} \left[\frac{\partial F_1}{\partial x} \delta x + \frac{\partial F_1}{\partial \dot{x}} \delta \dot{x} \right] dt + \int_{t^{s+}}^{t^f} \left[\frac{\partial F_2}{\partial x} \delta x + \frac{\partial F_2}{\partial \dot{x}} \delta \dot{x} \right] dt = 0 \tag{2. 221}$$

But, the second term in each integral can be integrated by parts as

$$\int \frac{\partial F}{\partial \dot{x}} \delta \dot{x} = \frac{\partial F}{\partial \dot{x}} \delta x - \int \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \right) \delta x \, dt \tag{2. 222}$$

Equation (2. 221) then becomes:

$$\delta J = \int_{t^o}^{t^{s-}} \left[\frac{\partial F_1}{\partial x} - \frac{d}{dt}\left(\frac{\partial F_1}{\partial \dot{x}}\right)\right] \delta x\, dt + \int_{t^{s+}}^{t^f} \left[\frac{\partial F_2}{\partial x} - \frac{d}{dt}\left(\frac{\partial F_2}{\partial \dot{x}}\right)\right] \delta x\, dt \qquad (2.223)$$

$$+ \left[F_1\, dt + \frac{\partial F_1}{\partial \dot{x}}\, \delta x \right]^{t^{s-}} - \left[F_2\, dt + \frac{\partial F}{\partial \dot{x}}\, \delta x \right]^{t^{s+}} = 0$$

The first two terms of Equation (2.223) are zero since they are merely the Euler-Lagrange equations. Now, since the staging time is the parameter that is to be varied, the state variation and the staging time variation are related by

$$\delta x = dx - \dot{x}\, dt \qquad (2.224)$$

Using Equation (2.224), Equation (2.223) becomes

$$\left[F_1\, dt + \frac{\partial F_1}{\partial \dot{x}} (dx - \dot{x}\, dt) \right]^{t^{s-}} - \left[F_2\, dt + \frac{\partial F_2}{\partial \dot{x}} (dx - \dot{x}\, dt) \right]^{t^{s+}} = 0 \qquad (2.225)$$

or

$$\left[\left(F_1 - \frac{\partial F_1}{\partial \dot{x}} \dot{x}\right) dt + \frac{\partial F_1}{\partial \dot{x}}\, dx \right]^{t^{s-}} = \left[\left(F_2 - \frac{\partial F_2}{\partial \dot{x}} \dot{x}\right) dt + \frac{\partial F_2}{\partial \dot{x}}\, dx \right]^{t^{s+}}$$

But, since $F_1^{s-} = F_2^{s+}$ this equation reduces to

$$\frac{\partial F^{(-)}}{\partial \dot{x}}\, dx^{(-)} - \frac{\partial F^{(+)}}{\partial \dot{x}}\, dx^{(+)} - (H^{(-)} - H^{(+)})\, dt = 0 \qquad (2.226)$$

where (+) has been used for t^{s+}
(-) has been used for t^{s-}
and H is the hamiltonian $\left(H = \frac{\partial F}{\partial \dot{x}}\, \dot{x}\right)$

Equation (2.226) represents the continuity that must exist through the staging point. However, for each particular problem, there is a different relationship between dX and dt at the staging point. Hence, Equation (2.226) dictates different conditions that must be met at the staging point for different missions.

In order to illustrate the use of Equation (2.226), several different cases will be considered. Each case will assume different continuity conditions during the staging point. It will be seen that Equation (2.226) will specify the different conditions that must be satisfied in each case.

Case 1

First consider the case of a vehicle configuration in which it is desired to burn the first stage such that staging occurs at a fixed time subject to no other constraints. If the drop-off weight is assumed to be the same for various trial cases of staging, Equation (2.226) reduced to

$$\frac{\partial F_1^{(-)}}{\partial \dot{x}} = \frac{\partial F_2^{(+)}}{\partial \dot{x}} \tag{2.227}$$

or

$$p_i^{(-)} = p_i^{(+)} \tag{2.228}$$

This condition results from the fact that all state variables must remain continuous across the staging point $(d x_i^{(-)} = d x_i^{(+)})$ and the drop-off weight is fixed $(d m^{(-)} = d m^{(+)})$

Case 2

If the staging time is free but the drop-off weight is fixed, then Equation (2.226) yields

$$\frac{\partial F^{(-)}}{\partial \dot{x}} = \frac{\partial F^{(+)}}{\partial \dot{x}} \tag{2.229}$$

or

$$p_i^{(-)} = p_i^{(+)}$$

and

$$H^{(-)} = H^{(+)} \tag{2.230}$$

Equation (2.29) results from the fact that $dx^{(-)} = dx^{(+)}$, since the state variables must be continuous (the drop-off mass is still fixed so $dm^{(-)} = dm^{(+)}$). Equation (2.230) results from fact that the staging time is left open so $dt \neq 0$ and the third term of Equation (2.226) is zero if Equation (2.230) is satisfied.

Case 3

As a third example, consider the case where the staging time is open but the drop-off weight is a function of the weight at the end of the burning of the first stage, i. e.,

$$m^{(+)} = m^{(-)} - m(m^{(-)}) \tag{2.231}$$

Again Equation (2.226) requires that

$$\frac{\partial F^{(-)}}{\partial \dot{x}} = \frac{\partial F^{(+)}}{\partial \dot{x}} \tag{2.232}$$

or

$$p_i^{(-)} = p_i^{(+)} \tag{2.233}$$

for all state variables except mass. However, the third term of Equation (2.236) requires that the hamiltonian be continuous at the staging point, i. e.,

$$H^{(-)} = H^{(+)} \tag{2.234}$$

The only remaining condition to be found is the requirement on the mass state variable during the staging point. Since the state variable for mass is not continuous and the drop-off weight is not fixed, the continuity required from Equation (2.226) is

$$\frac{\partial F^{(-)}}{\partial \dot{m}} dm^{(-)} = \frac{\partial F^{(+)}}{\partial \dot{m}} dm^{(+)} \tag{2.235}$$

or

$$p_m^{(-)} dm^{(-)} = p_m^{(+)} dm^{(+)} \tag{2.236}$$

where the relationship between $dm^{(-)}$ and $dm^{(+)}$ can be found from Equation (2.231) as

$$dm^{(+)} = dm^{(-)} - \frac{dm(m^{(-)})}{dm^{(-)}} dm^{(-)} \tag{2.237}$$

Combining Equations (2.236) and (2.237) yields

$$p_m^{(-)} = p_m^{(+)} \left[1 - \frac{1}{m^{(-)}} \quad \frac{dm(m^{(-)})}{dm^{(-)}} \right]$$

The same procedures shown in the previous examples can be used for most staging conditions. Equation (2.236) will yield various conditions to be satisfied during the staging point for different conditions that are specified at the staging point. These conditions provide additional equations that must be used to obtain the optimum solution in the analytical or numerical solution of the problem.

2.3 ORBITAL PROBLEMS

This section of the monograph is devoted to problems involving a vehicle initially in an orbit in a central force field. Two types of maneuvers will be considered for the vehicle. One is a transfer of orbit and the other is a rendezvous with another vehicle. The rendezvous problem could be considered as a special case of the orbital transfer problem with the difference being that an additional constraint is required in the rendezvous problem to place the vehicle at the correct position in the new orbit. As discussed here, however, the rendezvous problem will be formulated in a different manner than the orbital transfer problem. In the rendezvous problem the equations of motion will be referenced to the vehicle in the target orbit, and it will be assumed that the relative motion can be adequately described by linear equations. The use of such equations imposes the constraint that the vehicle be placed in the proper position in the target orbit. In contrast, the orbital transfer problem generally involves non-linear equations of motion; however, if the transfer is to be made between very nearly identical orbits, as might be the case if it were required to return a vehicle to some reference orbit from which it had strayed due to some uncompensated perturbative force, the equations of motion can be linearized. It will be shown in the following sections that such linear equations, in either the rendezvous or orbit transfer problem, do not necessarily mean that an analytic solution to the problem is available.

2.3.1 Terminal Rendezvous

"Terminal Rendezvous" as used in this section is the process of matching the position and velocity of one space vehicle, called the rendezvous vehicle, with that of another space vehicle, called the target vehicle. It is assumed that the orbits of the two vehicles are nearly the same and that the separation distance is small (Such conditions can be expected to occur as a result of boost guidance and/or midcourse guidance.) This assumption allows the equations of motion to be adequately represented by linear equations. To simplify the development only circular orbits are considered. The development of the equations for this situation is presented in the following section, and their use in the optimization of the rendezvous maneuver is discussed in subsequent sections.

2.3.1.1 Linear Equations of Motion

To develop the linear equations of motion consider a coordinate system centered at the target vehicle with the y axis along the radius vector from the central body and the x, y plane in the plane of the target orbit as shown in Figure 3.1

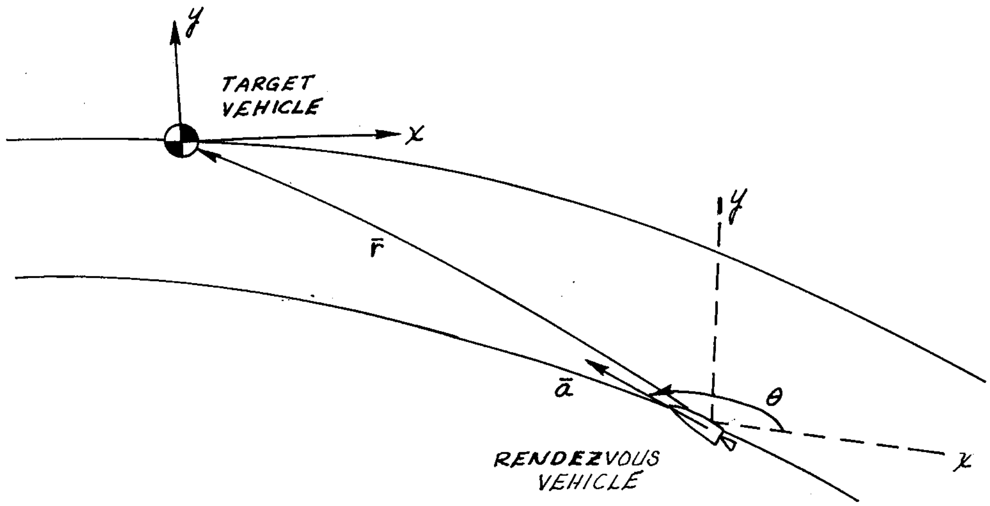


Figure 3.1 Rendezvous Geometry

The equations of motion of the rendezvous vehicle in this coordinate system are

$$\ddot{\underline{r}} = \Delta g - 2\underline{\omega} \times \underline{v} - \bar{\underline{\omega}} \times (\underline{\omega} \times \underline{r})$$

where

Δg = relative acceleration of gravity
ω = angular rate of target vehicle
$\underline{r}$ = relative position vector
$\underline{V}$ = relative velocity vector

The first step in the solution is to express Δg as a function of r

$$\Delta g = \frac{\partial g}{\partial \underline{r}} \Delta r$$

to accomplish this expansion, consider the component of gravity in the x direction; i.e.,

$$g_x = \frac{-\mu x}{(x^2 + y^2 + z^2)^{3/2}}$$

thus, forming the partial derivatives of g_x with respect to x, y, and z gives

$$\frac{\partial g_x}{\partial x} = -\frac{\mu}{r^3} + \frac{3\mu x^2}{r^5}$$

$$\frac{\partial g_x}{\partial y} = \frac{3\mu x y}{r^5}$$

$$\frac{\partial g_x}{\partial z} = \frac{3\mu x z}{r^5}$$

Similarly, the partial derivatives of g_y and g_z are obtained as

$$\frac{\partial g_y}{\partial x} = \frac{3\mu y x}{r^5} \qquad \frac{\partial g_z}{\partial x} = \frac{3\mu z x}{r^5}$$

$$\frac{\partial g_y}{\partial y} = -\frac{\mu}{r^3} + \frac{3\mu y^2}{r^5} \qquad \frac{\partial g_z}{\partial y} = \frac{3\mu z y}{r^5}$$

$$\frac{\partial g_y}{\partial z} = \frac{3\mu y z}{r^5} \qquad \frac{\partial g_z}{\partial z} = -\frac{\mu}{r^3} + \frac{3\mu y^2}{r^5}$$

Thus, the partials of gravity in an inertial system can be written as

$$\frac{\partial g}{\partial r} = \begin{bmatrix} \frac{-\mu}{r^3} + \frac{3\mu x^2}{r^5} & +\frac{3\mu x y}{r^5} & +\frac{3\mu x z}{r^6} \\ +\frac{3\mu x y}{r^5} & -\frac{\mu}{r^5} + \frac{3\mu y^2}{r^5} & +\frac{3\mu y z}{r^5} \\ +\frac{3\mu x z}{r^5} & +\frac{3\mu y^2}{r^5} & -\frac{\mu}{r^5} + \frac{3\mu z^2}{r^5} \end{bmatrix}$$

and the corresponding variation as

$$\Delta g = \left[\frac{\partial g}{\partial r}\right] \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix}$$

But for a circular orbit $\omega^2 = \mu/r^3$, and for the rotating coordinate system chosen, $x = z = 0$ and $y = r$. Thus, the matrix representing the variations of gravity in this system can now be written as

$$\frac{\partial g}{\partial r} = \begin{bmatrix} -\omega^2 & 0 & 0 \\ 0 & 2\omega^2 & 0 \\ 0 & 0 & -\omega^2 \end{bmatrix}$$

and the linearized equations of motion become

$$\ddot{x} = 2\dot{y}\omega$$

$$\ddot{y} = 3\omega^2 y - 2\dot{x}\omega$$

$$\ddot{z} = -\omega^2 z$$

Since the Z motion is uncoupled in the linear gravity expansion, it will be (temporarily) neglected in the equations. The use of equations with a linear gravity model do not necessarily yield an analytic solution to the rendezvous problem. Therefore, it is sometimes useful to consider the gravity free case. In this case, planar motion can be assumed and the dimensionality of the problem can be reduced. However, the forms of the state equations for both the linear and gravity free problem are the same as is demonstrated below. The equations of motion for the rendezvous vehicle (assuming no relative acceleration due to gravity) are

$$\ddot{x} = a \cos\theta$$
$$\ddot{y} = a \sin\theta \tag{3.1}$$
$$0 \le a \le a_{MAX}$$

where

a = magnitude of the thrust acceleration

θ = angle of thrust vector relative to x axis

a_{max} = maximum value of the thrust

Now, since optimizing procedures to be used have been formulated such that the state vector is constrained to obey a first order differential equation, the equations of Motion (3.1) will be written in that form by the definition of new state variables.

$$
\begin{aligned}
x_1 &\triangleq x \\
x_2 &\triangleq y \\
x_3 &\triangleq \dot{x} \\
x_4 &\triangleq \dot{y}
\end{aligned}
\tag{3.2}
$$

Using these definitions, the equations of motion become

$$
\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= a \cos \theta \\
\dot{x}_4 &= a \sin \theta
\end{aligned}
$$

This set can be written in matrix form as

$$\dot{\underline{x}} = A\underline{x} + f(a, \theta) \tag{3.3}$$

where

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad f(a,\theta) = \begin{bmatrix} 0 \\ 0 \\ a \cos \theta \\ a \sin \theta \end{bmatrix}, \text{ and } A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.4}
$$

Thus, in terms of the state variables defined by equation 3.2, the linear equations of motion are

$$\dot{\underline{x}} = A\underline{x} + \frac{\partial \underline{g}}{\partial \underline{x}} \underline{x} + f(a, \theta) \tag{3.5}$$

where

$$
\frac{\partial \underline{g}}{\partial \underline{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3\omega^2 & 0 & 0 & -2\omega \\ 0 & 2\omega & 0 & 0 \end{bmatrix}
$$

Equation (3.5) can now be written in the form

$$\underline{\dot{x}} = \tilde{A}\,\underline{x} + f(a, \theta) \qquad (3.6)$$

where

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & -2\omega \\ 0 & 2\omega & 0 & 0 \end{bmatrix} \qquad (3.7)$$

If the target orbit is circular then the mean motion, ω , will be a constant and both Equation (3.3) and (3.6) are first order, linear differential equations with constant coefficients. If the target orbit is nearly circular, the matrix $\tilde{A}$ will be time varying and Equation (3.6) will require different techniques to obtain a solution. The techniques for obtaining solutions to Equations of the form (3.3) and (3.6) are discussed in Reference (3.1).

In summary, this section has demonstrated that the state equations for a rendezvous problem in rectangular coordinates centered at the target vehicle have the form

$$\underline{\dot{x}} = A\,\underline{x} + f(a, \theta) \qquad (3.8)$$

for both a constant and linear gravity model.

2.3.1.2 Solution of the State and Co State Equations

The techniques required to obtain solutions to equations of the form of (3.8) have been discussed in detail in a previous monograph (Reference (3.1)) and, therefore, will not be discussed here. However, since the solutions of equations of the form of (3.8) are used in subsequent sections the form of the solution equations will be summarized below.

The solution to the homogenous state equation [i.e., $f(a, \theta)$ set equal to zero in Equation (3.8)]

$$\dot{x} = A\,x \qquad (3.9)$$

is given by the equation

$$x(t) = \phi(t, t_0)\, x(t_0) \qquad (3.10)$$

where $\phi(t, t_0)$ is the "transition matrix" for the system. The solution of the corresponding co-state equation

$$\dot{p} = -A^T p \tag{3.11}$$

is

$$p(t) = [\phi^{-1}(t, t_0)]^T p(t_0) \tag{3.12}$$

or

$$p(t) = \phi^T(t_0, t)\, p(t_0) \tag{3.13}$$

where the second form is obtained from the identity

$$\phi^{-1}(t, t_0) = \phi(t_0, t)$$

For the matrix A of Equation (3.4), the transition matrix has the form

$$\phi(t, t_0) = \begin{bmatrix} 1 & 0 & (t-t_0) & 0 \\ 0 & 1 & 0 & (t-t_0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

and for the matrix A of Equation (3.6), the form is

$$\phi(t,t_0) = \begin{bmatrix} 2-\cos\omega\tau & \sin\omega\tau & \dfrac{\sin\omega\tau}{\omega} & \dfrac{2(1-\cos\omega\tau)}{\omega} \\ 2\sin\omega\tau - 3\omega\tau & 2\cos\omega\tau - 1 & \dfrac{2(\cos\omega\tau - 1)}{\omega} & \dfrac{4\sin\omega\tau - 3\omega\tau}{\omega} \\ \omega(3\omega\tau - \sin\omega\tau) & \omega(1-\cos\omega\tau) & 2-\cos\omega\tau & 3\omega\tau - 2\sin\omega\tau \\ \omega(\cos\omega\tau - 1) & -\omega\sin\omega\tau & -\sin\omega\tau & 2\cos\omega\tau - 1 \end{bmatrix} \tag{3.15}$$

where

$$\tau = t - t_0$$

The complete solution of Equation (3.8) is given by

$$x(t) = \phi(t, t_o)\, x(t_o) + \int_{t_o}^{t} \phi(t, \tau)\, f(a, \theta)\, d\tau \qquad (3.16)$$

2.3.1.3 Minimum Time Rendezvous

The problem of achieving a rendezvous in the least amount of time is discussed in this section. The problem is formulated first under the assumption that the fuel supply is unlimited and then under the assumption that there is a fixed amount of fuel available. These two conditions lead to different types of control programs as is demonstrated below. For the minimum time problem, the fact that the system equations are linear is not sufficient to allow an analytical solution and one numerical technique which can be used to obtain a solution is discussed and some results using this method presented.

2.3.1.3.1 Optimization Using the Pontryagin Maximum Principle. The cost function for a problem in which it is desired to achieve rendezvous in a minimum time is the terminal time.

$$\phi(x_f, t_f) = t_f \qquad (3.17)$$

and the terminal constraints which are to be satisfied are that the components of the state vector are zero.

$$\psi_i(\underline{x}_f) = x_i = 0 \qquad (3.18)$$

Thus, using equation (3.8) to represent the dynamics of the system, the hamiltonian is

$$H = P^T A x + P_3\, a \cos\theta + P_4\, a \sin\theta \qquad (3.19)$$

But, according to the Maximum Principle, the hamiltonian must be maximized with respect to the control variables. Thus, maximizing H with respect to θ gives

$$\frac{dH}{d\theta} = -a P_3 \sin\theta + a P_4 \cos\theta = 0$$

or

$$\tan\theta = \frac{P_4}{P_3} \qquad (3.20)$$

Now, since the thrust magnitude is bounded by zero and a_{MAX} and since (Equation (3.19)) it is seen that the sign of the coefficient of a determines which bound that a assumes, the coefficient of a is called the switching function S. From Equation (3.19), this function is

$$S = P_3 \cos\theta + P_4 \sin\theta$$

The control variable θ can be eliminated from this equation by the use of (3.20) to yield the magnitude of S as

$$S = \sqrt{P_3^2 + P_4^2}$$

Now, the control variable a is given by

$$a = \begin{cases} a_{MAX} & S > 0 \\ 0 & S < 0 \end{cases} \tag{3.21}$$

Equation (3.21) indicates that the thrust magnitude varies in a bang-bang fashion, i.e., it is either full on or zero with no intermediate throttle settings. It turns out, however, that there are no periods of zero thrust for the time optimal case, i.e., the positive sign for the square root in the switching function is always chosen. This fact can be demonstrated by taking the second derivative of the hamiltonian with respect to θ so that **the quadrant is determined by rewriting the control vector in a different but equivalent form. If the change of variables**

$$u_1 = a \cos\theta$$

$$u_2 = a \sin\theta$$

is made, then the state equation (equation (3.8)) can be rewritten as

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{3.22}$$

where

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.23}$$

and where the control is constrained by the relation

$$\|\underline{u}\| \leq a_{MAX}$$

For the state equation of (3.22), the hamiltonian is

$$H = P^T A\underline{x} + P^T B\underline{u} \tag{3.24}$$

and maximization of the hamiltonian requires that the term $P^T Bu$ be maximum. This term is the inner product of the vector $B^T P$ and the vector $\underline{u}$, i.e.

$$P^T B u \equiv \langle B^T P, u \rangle$$

and since this inner product can be interpreted as the projection of $\underline{u}$ in the direction of $B^T P$, it is clear that this projection will be maximum if $\underline{u}$ is in the direction of $B^T P$ with its largest possible magnitude. i.e.,

$$\underline{u} = a_{MAX} \frac{B^T P}{\| B^T P \|} \qquad (3.25)$$

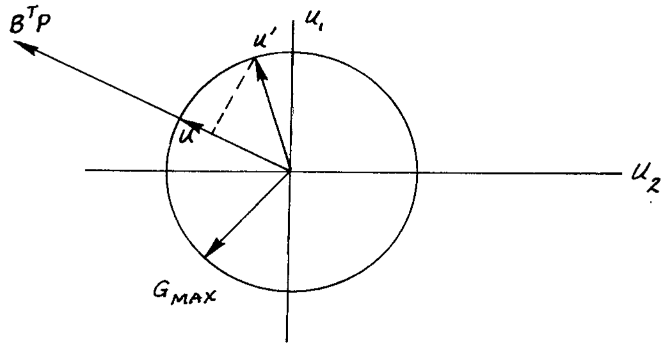


FIGURE 3.2 Graphical Description of Maximizing $P^T Bu$

Note that since the co-state vector $\underline{P}$ is given in terms of the transition matrix (Reference (3.1)), it is continuous and equation (3.25), therefore, indicates the magnitude of the control vector $\underline{u}$ does not change discontinuously from zero to A_{max} but rather has the constant value of A_{max}. However, determination of the co-state vector in order to define the control (Equation (3.20) requires an iterative process since the initial conditions on the co-state are unknown.

If the gravity-free rendezvous is being considered, the transition matrix is applicable; the integration indicated on the right-hand side of Equation (3.18) can then be accomplished (see reference 3.1, page 81) and an analytic, though transcendental, equation involving the unknown co-state initial conditions obtained. Then, since boundary conditions for the state vector $\underline{X}$ are known at both ends of the trajectory, the initial co-state values can be determined. On the other hand, if the linear gravity model is used, then the transition matrix of equation (3.15) is required. In this case, the integration in equation (3.16) cannot be performed analytically and an expression relating the initial values of the co-state vector to the boundary values of the state vector cannot be found. The solution to this problem must be obtained by numerical methods such as the one described in section 2.3.1.3.2.

If it is desired to study time optimal rendezvous under the constraint that a limited amount of fuel (or alternately a limit on the total velocity change ΔV) is available, the inclusion of an additional state variable representing the instantaneous fuel mass is necessary. This inclusion first requires the development of the equation governing the instantaneous fuel from the rocket thrust equations. If the rocket exhaust leaves the nozzle at a constant velocity c (relative to the rocket) and if the rocket is moving with velocity v , the rate of change of momentum (p) of the rocket is

$$\frac{d(p_e)}{dt} = \dot{m}v + ma$$

The rate of change of momentum of the exhaust gas is

$$\frac{d(p_g)}{dt} = \dot{m}(c - v)$$

Now, since the total momentum of the system is constant, the time rate of change of momentum of the exhaust gas must equal the time rate of change of momentum of the rocket. Thus, equating the expressions for these quantities gives

$$\dot{m}c = ma \tag{3.26}$$

where

$\dot{m}$ = mass flow rate

c = rocket exhaust velocity

a = rocket acceleration

m = instantaneous rocket mass

and where the instantaneous fuel mass is given by the expression

$$m_p(t) = m_{p_0} - \int_{t_0}^{t} \dot{m}(t)\,dt$$

or

$$m_p(t) = m_p - \int_{t_0}^{t} \frac{m(t)\,a(t)}{c}\,dt$$

Thus, if it is assumed that the mass of the fuel is only a small fraction of the mass of the rocket, the term $m(t)$ under the integral sign can be replaced by a constant to yield

$$m_p(t) - m_{p_0} = -\frac{m}{c}\int_{t_0}^{t} a(t)\,dt = -\frac{m}{c}\Delta V$$

Equation (3.26) is the state equation for the mass flow, and this equation indicates the equivalence of a fixed amount of fuel to a fixed velocity change.

The minimization is now to be performed subject to the constraint

$$\left.\begin{aligned} & m(t_0) - m(t_f) \leq m_p \\ \text{or} \quad & \\ & \int_{t_0}^{t_0} a \, dt \leq \Delta V \end{aligned}\right\} \tag{3.27}$$

where m_p is the total propellant mass and

$$\Delta V = \frac{c m_p}{m}$$

This is accomplished by first performing the minimization assuming there is no propellant constraint. If the result of this minimization indicates that the constraint (3.27) has not been violated, then this result is the desired solution. However, if the final mass does not fall within the bound given by (3.27), the minimization must be repeated with the inequality of (3.27) replaced by equality and adjoined to the hamiltonian. For example, in the two-dimensional problem, the components of the state vector are given by (3.2). Thus, adding the additional component corresponding to the amount of fuel results in the state vector

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \begin{aligned} x_1 &= x \quad ; & x_4 &= \dot{y} \\ x_2 &= y \quad ; & x_5 &= m \\ x_3 &= \dot{x} \end{aligned}$$

where the boundary condition on X_5 is

$$x_5(0) = m_p \qquad x_5(t_f) = 0$$

One interesting result of the fuel constraint is that the thrust now varies discontinuously in a bang-bang fashion, and the solution allows periods of coasting, as opposed to the unconstrained case in which the thrust is always full on. This result will be demonstrated in figures which are presented in the next section.

2.3.1.3.2 Application of Neustadt's Method to Minimum Time Rendezvous. To summarize the results of the minimum time problem, the pertinent equations are listed below. First, the state equations,

$$\dot{x} = Ax + f(a, \theta)$$

the co-state equations,

$$\dot{p} = -A^T p$$

and the control equations,

$$a = a_{MAX}$$

$$\theta = \arctan \frac{p_4}{p_3}$$

But, the co-state equations are linear with a solution of the form

$$P(t) = \Phi(t, t_o) P_o$$

Therefore, if P_o were know, the control would be known as a function of time and the problem would be solved. However, P_o is not known, nor for that matter is $P(t_f)$. However, the state equations can be written in terms of the co-state variables by substitution of the control equations in the state equations. If this substitution is made, the set of n state equations and n co-state equations constitute a set of $2n$ first order differential equations. Since the state vector is known both at the initial time and at the terminal time there are $2n$ boundary conditions. This information is sufficient to obtain a solution; however, the set is of the non-linear two-point boundary value type and, as such, must be solved by numerical techniques. One such numerical technique for solving the two point boundary value problem encountered in the investigation of minimum time problems in general has been developed by L. Neustadt (Reference (3.4)).

Neustadt's method [References (3.3) and (3.4)] is concerned with finding the correct initial value for the co-state vector P(o) so that P(t) can be found from Equation (3.12). Neustadt showed that the correct value of P(o) maximized a certain function (to be developed below) and thus reduced the solution of the two-point value problem to the solution of an ordinary maximization problem in several variables. To begin the development, consider the solution for the state vector [Equation (3.16)].

$$\underline{x}(t) = \Phi(t, t_o)\, \underline{x}(t_o) + \int_{t_o}^{t} \Phi(t, \tau)\, \underline{f}(a, \theta)\, d\tau \qquad (3.28)$$

At the time of rendezvous (call this time t_f), the state vector on the left of (3.28) will be zero, since that is the required boundary condition. Using this fact (3.28) can be rewritten at $t = t_f$ as

$$\underline{x}(t_o) = -\int_{t_o}^{t_f} \Phi(t_o, \tau)\, \underline{f}(a, \theta)\, d\tau \qquad (3.29)$$

Now, for any particular initial state vector $\underline{X}(t_o)$ the control variables, a and θ , are chosen so that (3.29) is valid for the smallest t_f (this solution corresponds to choosing the correct initial co-state vector). If all possible values of the initial co-state vector, P(o), and an arbitrary time, t, are considered, then equation (3.29) can be thought of as defining the set of all possible initial conditions which would produce a rendezvous in time t. This set will be indicated as $\underline{X}_o$ (P_o, t) as a reminder that any particular member of the set is a function of the initial co-state vector P_o and t. Equation (3.28) defines this set, i.e.,

$$\underline{x}_o(P_o, t) = -\int_{t_o}^{t} \Phi(t_o, \tau)\, \underline{f}(P_o)\, d\tau \tag{3.30}$$

where the control function $\underline{f}(a, \theta)$ has been written as a function of P_o to illustrate the dependence on P_o. As a final word of explanation of equation (3.30), note that the initial condition to the original problem $\underline{X}$ (t_o) is a member of the set $\underline{X}$ (P_o^*, t), t $\geqslant$ t_f, where P_o^* is the correct initial co-state vector. Suppose an arbitrary initial co-state vector, say $\underset{\sim}{P}_o$, is selected and the inner product of this vector and $\underline{X}_o$ ($\underset{\sim}{P}_o$, t) is formed.

$$\underset{\sim}{P}_o^T\, \underline{x}_o(\underset{\sim}{P}_o, t) = -\int_{t_o}^{t} \underset{\sim}{P}_o^T \Phi(t_o, \tau)\, f(P_o)\, d\tau \tag{3.31}$$

But, the first two factors under the integral are P(τ) (from equation (3.12), so (3.31) becomes

$$\underset{\sim}{P}_o^T\, x_o(\underset{\sim}{P}_o, t) = -\int_{t_o}^{t} \underset{\sim}{P}^T(\tau)\, f(\underset{\sim}{P}_o)\, d\tau \tag{3.32}$$

Now, the integrand is that part of the hamiltonian (see equation (3.24) which was required to be maximum, thus implying that the inner product on the left of (3.32) is less than the inner product of $\underset{\sim}{P}_o$ and any other member of the set X_o(P_o, t), i.e.,

$$\underset{\sim}{P}_o^T\, x_o(\underset{\sim}{P}_o, t) < \underset{\sim}{P}_o^T\, x_o(P_o, t), \qquad P_o \neq \underset{\sim}{P}_o$$

In particular, if the initial state vector of the problem, $\underline{X}$ (t_o), is a member of the set (i.e., if t $\geqslant$ t_f), then

$$\underset{\sim}{P}_o^T\, x_o(P_o, t) \leq P_o^T\, \underline{x}(t_o)$$

or

$$\underset{\sim}{P}_o^T \left[x_o(\underset{\sim}{P}_o, t) - x(t_o) \right] \leq 0 \tag{3.33}$$

Let P_o be chosen so that $P_o^T X(t_o) \le o$. and consider the case when $\underline{X}_o(P_o, t) = X(t_o)$. Then at $t = o$, the inner product on the left of (3.33) will be **positive** but at $t \ge t_f$ the inequality (3.33) must hold; therefore, there is some time, say t^*, at which the inner product $\underset{\sim}{P}_o^T[x(\underset{\sim}{P}_o, t) - x_o]$ is zero. Furthermore, this time is bounded by zero and t_f, i.e., $0 \le t^* \le t$, and the right-hand equality will hold only if $x(\underset{\sim}{P}_o, t) = x(t_o)$. Therefore, the problem is now to find the maximum t^* as a function of P_o subject to the constraint

$$P_o^T\left[x_o(P_o, t) - x(t_o)\right] = 0$$

Figure 3.3 may aid in the clarification of **this discussion.**

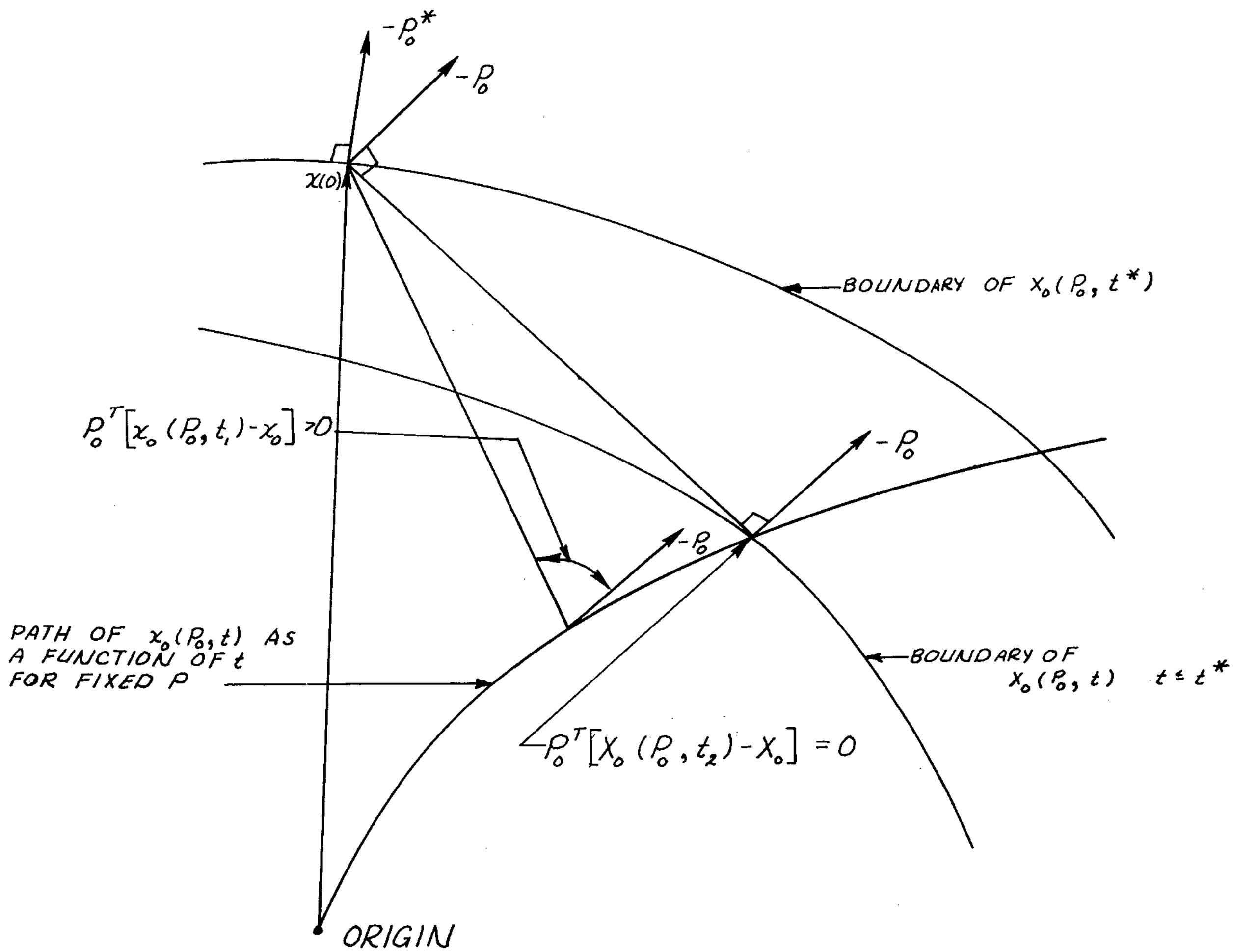


FIGURE 3.3 Geometric Description of the Maximization Process

In order to locate the value of P_o which maximizes t, it is necessary to proceed in the direction of the gradient of t. That is, for some particular value of P_o, there is corresponding value for t and $\partial t / \partial \underset{\sim}{P}_o$. If a change is to be made in P_o such that the new value of t is larger than the previous value, the increment ΔP_o should be taken in the direction of increasing t or (for maximum increase) along $\partial t/\partial P_o$. For example, suppose P_o had only two components, then the curve of constant $t = t(P_o)$ could be plotted in $P_1 \times P_2$ space. Now, the gradient of t is normal to these curves; therefore, the direction of a change in $\underline{P}_o$ to maximize t would be normal to a constant t curve (See Figure 3.4 below).

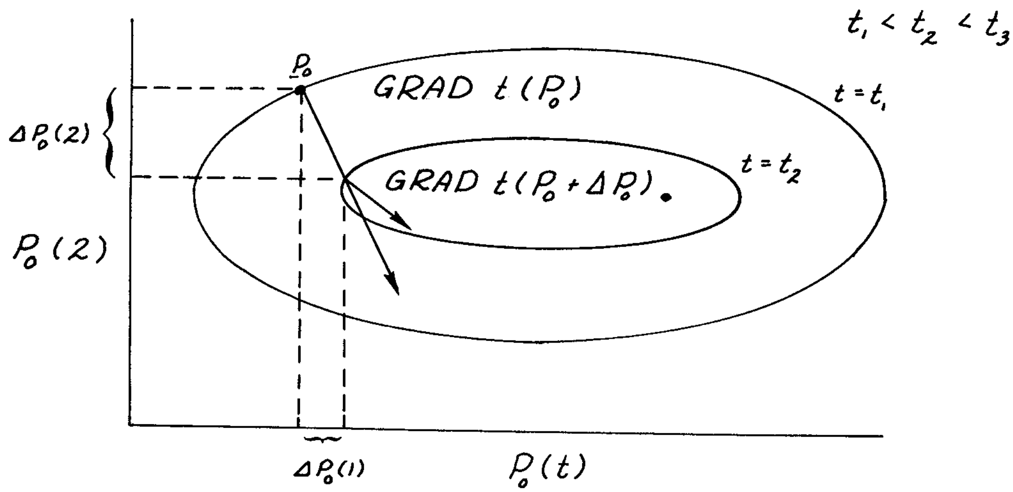


FIGURE 3.4 Determination of Maximum t

Thus, if $\underline{P}_o(n)$ represents the nth iteration of P_o, the iteration process is described by the equation

$$\underline{P}_o(n+1) = \underline{P}_o(n) + k \frac{\partial t}{\partial P_o}$$

where k is a constant which determines the step size. In reference 3.4 it is shown that the direction of $\partial t/\partial P_o$ is given by the vector

$$[x_o(P_o, t) - x(t_o)]$$

so that the gradient has a particularly simple form once the quantity has been computed. A summary of Neustadt procedure is given below.

1. Guess Po such that $P_o^T X_o < 0$
2. Compute $\underline{X}_o(P_o, t)$ from equation (3.29) as a decreasing function of time. Stop the calculation when

$$P_o^T\left[x_o(P_o, t) - x(t_o)\right] = 0$$

3. For the next estimate of P_o by multiplying the quantity $(X_o(P_o, t) - X(t_o))$ by a constant k (the constant k is the stepsize and its magnitude must be chosen so that the linearity is preserved), and subtract from the previous P_o i.e.

$$P_o(n+1) = P_o - k\left[x_o(P_o, t) - x(t_o)\right].$$

4. If the process has not converged, repeat steps two and three.

Paiewonsky and Woodrow (Reference 3.3) used Neustadt's method to obtain solutions to a three-dimensional time optimal rendezvous problem with a fuel constraint similar to that described in section 2.3.1.3.1. The results indicated that it is possible to obtain the solution of the two-point boundary value problem in a reasonable number of iterations (at least for the sets of initial conditions studied). Figure 3.5 below shows an example of the convergence of the process using Powell's method (see Reference 3.3) to locate the maximum t*. The initial guess for P_o for this case (as for all cases studied in the report) is a unit vector parallel to $\underline{X}(t_o)$).

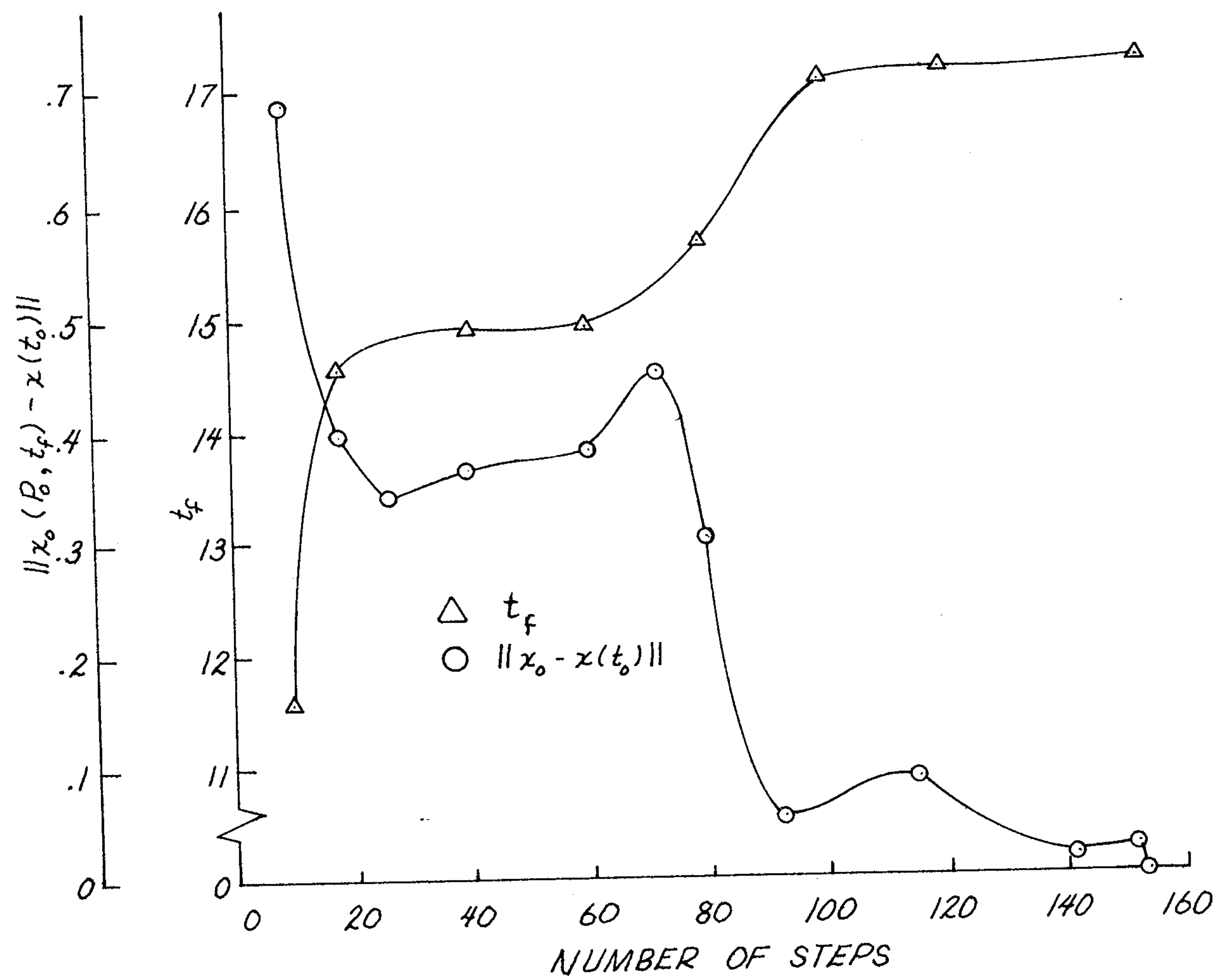


FIGURE 3.5 Convergence of the Numerical Calculation

Figure 3.6 illustrates the burning schedule as a function of time with the maximum velocity change (the total fuel available is related to the maximum velocity change by equation 3.26) as a parameter. It can be seen from this figure, as was mentioned in section 2.2.3.1.2, that the addition of a fuel constraint results in coast periods during the maneuver.

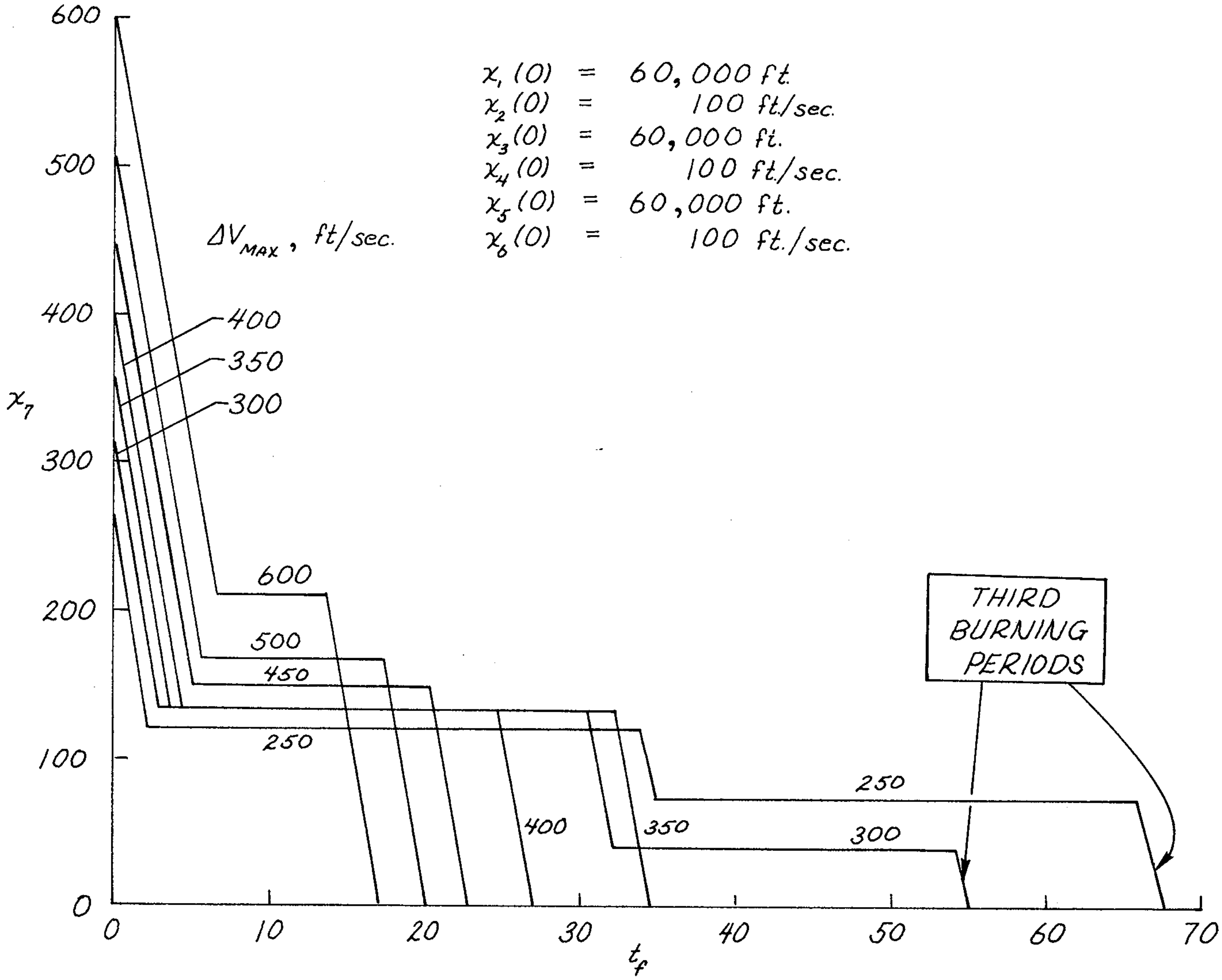


FIGURE 3.6 Fuel Time History for Time Optimal Rendezvous Maneuver

2.3.1.4 Penalty Function and Quadratic Cost in the Rendezvous Problem

This section presents as an example of the application of Dynamic Programming and the use of a penalty function to achieve rendezvous , a review of the work of F. T. Smith (Reference 3.6). The main difference between this development and earlier approaches is in the definition of the state. Smith requires the state to contain components of the absolute (as opposed to relative) motion of the target vehicle and the rendezvous vehicle*, i.e., if X_T is a vector describing the position and velocity of the target vehicle in an inertial coordinate system and X_R is a vector defining the position and velocity of the rendezvous vehicle in the same coordinate system, then the state vector, Z, used by Smith is

$$Z = \begin{pmatrix} x_T \\ \hline x_R \end{pmatrix} \tag{3.34}$$

This definition of the state vector is used to simplify the equations for the nonlinear gravity model and to facilitate the analysis of the effects of noise (The effects of noise will not be discussed here.) However, since the results of the deterministic problem presented in the reference uses the equations developed below, a derivation in terms of the state vectors used previously in the rendezvous problem will be made. After the presentation of the dynamic programming formulation of the problem, presented in section 2.3.1.4.2, a formulation using the maximum principle is presented in section 2.3.1.4.3 for the purpose of comparing these two techniques.

2.3.1.4.1 The Cost and Penalty Functions. In previous sections, the problem of driving the state vector to zero (i.e., achieving rendezvous) while minimizing some performance function, such as the final time, or the integral of the magnitude of the control, was considered. This section is concerned with minimizing a function of the expended control and the terminal state vector for the case in which there are no terminal constraints. In this problem, it is no longer required that the state vector be driven to zero only that its final magnitude be minimized. This method of formulation is called the penalty function approach, and is employed since it can allow a solution without iterations. The fact that an explicit solution can be obtained is the result of two implicit assumptions. The first, just noted, is that a terminal state small enough to satisfy the mission can be obtained without imposing a rigorous constraint. The second is that the performance of the system can be modeled satisfactorily by considering the integral of the square of the mangitude of the control rather than the integral of the

* A different development of this material will be presented to conform with notation used earlier.

magnitude of the control in the performance function. Under these assumptions, the total cost function to be considered is

$$J = \underline{x}^T(t_f)\,\underline{x}(t_f) + \lambda \int_{t_0}^{t_f} \underline{u}^T(\tau)\,\underline{u}(\tau)\,d\tau \tag{3.35}$$

where t_f is assumed to be known, and where the factor λ is a weighting factor whose magnitude must be adjusted to provide the desired performance. For example, suppose that a value of λ is picked at random and the optimization performed, but suppose the resulting minimum value of the state vector is unacceptably large. The value of λ is now decreased so that the cost of the terminal error is increased relative to the control cost and the optimization process repeated. In this way, a value for λ can be found which will result in a successful rendezvous, and the amount of control (i.e., fuel) used in the process will be approximately minimized.

2.3.1.4.2 Dynamic Programming Formulation. The minimum value of the cost function of equation (3.35) will be a function of the final time and the initial value of the state vector X_0. However, it will be convenient to assume that the final time is fixed and that the initial time is a variable so that the minimum value of the cost function is defined as

$$R(x_0, t_0) \triangleq \min_{u} \left\{ \underline{x}_f^T \underline{x}_f + \lambda \int_{t_0}^{t_f} u^T u \, d\tau \right\} \tag{3.36}$$

Now, suppose that the optimum control for the time interval $t_0 \leq t \leq t_0 + \Delta t$ is known and that application of this control over that interval results in the state $X_0 + \Delta X$. The principle of optimality requires that the remaining continuous choice of u in the time interval $t_0 + \Delta t \leq t \leq t_f$ be optimal with respect to the new initial state, i.e., u must be chosen to satisfy

$$\min_{u} \left\{ \underline{x}_f^T \underline{x}_f + \lambda \int_{t_0+\Delta t}^{t_f} \underline{u}^T \underline{u} \, d\tau \right\}$$

But this expression is the minimum cost defined by equation (3.36) with new arguments, i.e.,

$$R(x_0+\Delta x,\ t_0+\Delta t) = \underset{u}{MIN}\left\{ x_f^T x_f + \lambda \int_{t_0}^{t_f} u^T u \, d\tau \right\} \qquad (3.37)$$

The integral in (3.36) can be broken into the sum of an integral over the time period $t_0 \leq t \leq t_0 + \Delta t$ and another for the time period $t_0 + \Delta t \leq t \leq t_f$ But, the integral over $t_0 \leq t \leq t_0 + \Delta t$ can be approximated by a rectangular integration scheme so that equation (3.36) can be rewritten as

$$R(x_0, t_0) = \underset{u}{MIN}\left\{ \lambda u^T u \Delta t + R(x_0+\Delta x,\ t_0+\Delta t) \right\} \qquad (3.38)$$

Now, the minimum cost function of equation (3.37) can be expanded in a Taylor series as

$$R(x_0+\Delta x,\ t_0+\Delta t) \approx R(x_0, t_0) + \left(\frac{\partial R}{\partial x}\right)^T \frac{dx}{dt}\Delta t + \frac{\partial R}{\partial t}\Delta t$$

and the result substituted into (3.38) to yield

$$R(x_0, t_0) = \underset{u}{MIN} \quad \lambda u^T u \Delta t + R(x_0, t_0) + \left(\frac{\partial R}{\partial x}\right)^T \frac{dx}{dt}\Delta t + \frac{\partial R}{\partial t}\Delta t \Big\}$$

But, the term $R(x_0, t_0)$ in the brackets is not a function of u and is unaffected by the min operation; therefore, it can be taken out of the bracket to cancel the identical term on the left of the equal sign. Now noting that the factor Δt is arbitrary and that the factor $\frac{dx}{dt}$ can be replaced by its equivalent form from equation (3.22), there results

$$0 = \min_{u} \left\{ \lambda u^T u + \left(\frac{\partial R}{\partial x}\right)^T (Ax + Bu) + \frac{\partial R}{\partial t} \right\} \tag{3.39}$$

Thus, performing the minimization gives

$$u = -\frac{1}{2\lambda} B^T \frac{\partial R}{\partial x} \tag{3.40}$$

Substituting this equation in 3.39 gives

$$\frac{1}{4\lambda} \left(\frac{\partial R}{\partial \underline{x}}\right)^T BB^T \frac{\partial R}{\partial \underline{x}} + \left(\frac{\partial R}{\partial x}\right)^T \left(\frac{1}{2\lambda} BB^T \frac{\partial R}{\partial x} + Ax \right) + \frac{\partial R}{\partial t} = 0$$

or

$$\frac{\partial R}{\partial t} + \left(\frac{\partial R}{\partial x}\right)^T Ax - \frac{1}{4\lambda} \left(\frac{\partial R}{\partial x}\right)^T BB^T \frac{\partial R}{\partial x} \tag{3.41}$$

This partial differential equation can be solved by assuming a solution of the form

$$R(t, x) = x^T K x + x^T g + \phi(t) \tag{3.42}$$

where K is symmetric. Thus, the partial derivatives required in equation (3.41) are

$$\frac{\partial R}{\partial t} = x^T \dot{K} x - x^T \dot{g} + \dot{\phi}$$

$$\frac{\partial R}{\partial x} = 2Kx - g \tag{3.43}$$

Substituting these partial derivatives in Equation (3.41) gives

$$x^T\dot{K}x - x^T\dot{g} + \phi + (2Kx - g)^T Ax - \frac{1}{4\lambda}(2Kx - g)^T BB^T(2Kx - g) = 0$$

or

$$\left[\dot{\phi} - \frac{1}{4\lambda} g^T BB^T g\right] + x^T\left[-\dot{g} + A^T g + \frac{1}{2\lambda} KBB^T g\right] + x^T\left[\dot{K} + 2KA - \frac{1}{\lambda}K^T BB^T K\right]x = 0$$

Now, equating the coefficients of comparable powers of X on both sides of the equation yields

$$\begin{aligned} \dot{\phi} &= \frac{1}{4\lambda} g^T BB^T g \\ \dot{g} &= A^T g + \frac{1}{2\lambda} KBB^T g \\ \dot{K} &= -KA - A^T K + \frac{1}{\lambda} K^T BB^T K \end{aligned} \tag{3.44}$$

where the identity

$$X^T\left(2KA\right)X \equiv X^T\left(KA + KA + A^TK - A^TK\right)X \equiv X^T\left(KA + A^TK\right)X$$

has been used in the equation for $\dot{K}$. The first equation of the set (3.44) can be neglected because the control, given by Equation 3.40, does not involve the function ϕ. Boundary conditions for the remaining equations may be obtained by noting that the expression for the minimum cost at the final time, t_f, is $R(X_f, t_f) = X^T_f X_f$. Comparing this expression with equation (3.42) indicates the boundary conditions are

$$\begin{aligned} K(t_f) &= I \\ g(t_f) &= 0 \\ \phi(t_f) &= 0 \end{aligned}$$

Since the differential equation for $g(t)$ [the second equation of (3.43)] is homogenous, the boundary condition results in the function being identically zero for all t. Thus, only a solution for $K(t)$ need be obtained to determine the control.

If the matrix A is time invariant, as for a circular orbit, then an analytic solution for the control is possible. Otherwise a numerical solution for $K(t)$ can be obtained since the boundary conditions are known. For the case of a time invariant system, the differential equation which $K(t)$ obeys is a nonlinear differential equation of the Riccati type, and a solution can be determined in terms of the solutions of the associated equations

$$\frac{dY}{dt} = AY - \frac{1}{\lambda} BB^T Z$$

$$\frac{dZ}{dt} = -A^T Z \tag{3.45}$$

Thus, the solution for $K(t)$ in terms of the variables Y and Z is

$$K(t) = Z(t)\, Y^{-1}(t) \tag{3.46}$$

as may be verified by direct substitution. Before obtaining a solution to 3.45, it will be convenient to make a change of variables to facilitate introduction of the boundary condition on $K(t)$. Let $\tau = t_f - t$ so that equations 3.45 become

$$\frac{dY}{d\tau} = -AY(\tau) + \frac{1}{\lambda} BB^T Z(\tau)$$

$$\frac{dZ}{d\tau} = A^T Z(\tau) \tag{3.47}$$

Now, the solution for $Z(\tau)$ is

$$Z(\tau) = e^{A^T \tau} Z(0) \tag{3.48}$$

and, using this solution the solution for $Y(\tau)$ is

$$Y(\tau) = e^{-A^T \tau} Y(0) + \frac{1}{\lambda} \int_0^{\tau} e^{-A(\tau - s)} BB^T e^{A^T s} Z(0)\, dS$$

Thus, defining $H(\tau)$ as

$$H(\tau) = \int_0^\tau e^{AS} B B^T e^{AS} dS$$

the solution for $Y(\tau)$ can be written as

$$Y(\tau) = \frac{e^{-A\tau}}{\lambda}\left[\lambda Y(0) + H(\tau) Z(0)\right] \tag{3.49}$$

and substitution of (3.48) and (3.49) into (3.46) gives the solution for $K(\tau)$ as

$$K(t_f - \tau) = \lambda e^{A^T\tau} Z(0)\left[\lambda Y(0) + H(\tau) Z(0)\right]^{-1} e^{A\tau}$$

Now, at $\tau = 0$ this equation is (using the boundary condition $K(t_f) = I$)

$$K(t_f) = I = Z(0)\, Y^{-1}(0)$$

However, since there is only one boundary condition either $Z(0)$ or $Y(0)$ can be chosen arbitrarily. Therefore, let $Z(0) = Y(0) = I$ so that $K(t)$ becomes

$$K(t) = \lambda e^{A^T(t_f - t)}\left[\lambda I + H(t_f - t)\right]^{-1} e^{A(t_f - t)} \tag{3.50}$$

Finally, using (3.40), (3.41) and (3.50), the control can be written as

$$u(t) = -B^T e^{A^T(t_f - t)}\left[\lambda I + H(t_f - t)^{-1} e^{A^T(\tau - t)} x(t)\right] \tag{3.51}$$

In reference 3.6 this control was applied to the situation in which a target satellite was in a 300 mi earth circular orbit and the rendezvous vehicle was initially in a co-planar elliptical orbit such that a collision would occur at the apogee of this orbit at $t = t_f$ (its eccentricity was .024.) In this example, a value of λ of 10^{-3} was specified. The results obtained in the reference are presented below for several final times. These estimates of the expended control must be contrasted with that corresponding to the impulse required to match the velocities at apogee of .031. Since the discrepancy is large, more optimum weighting is suggested. Note that the result of the present weighting is an extreme cost for relatively small position errors.

Table 3.1 Results of Rendezvous Optimization

Duration (Min.)	Miss Distance (ft)	Velocity Error (ft/sec)	Propellant Mass Fraction
1.0	1.046	.000044	.1554
2.0	.260	.000056	.1434
3.0	.380	.000059	.1435
4.0	.328	.000004	.1435
5.0	.065	.000179	.1435

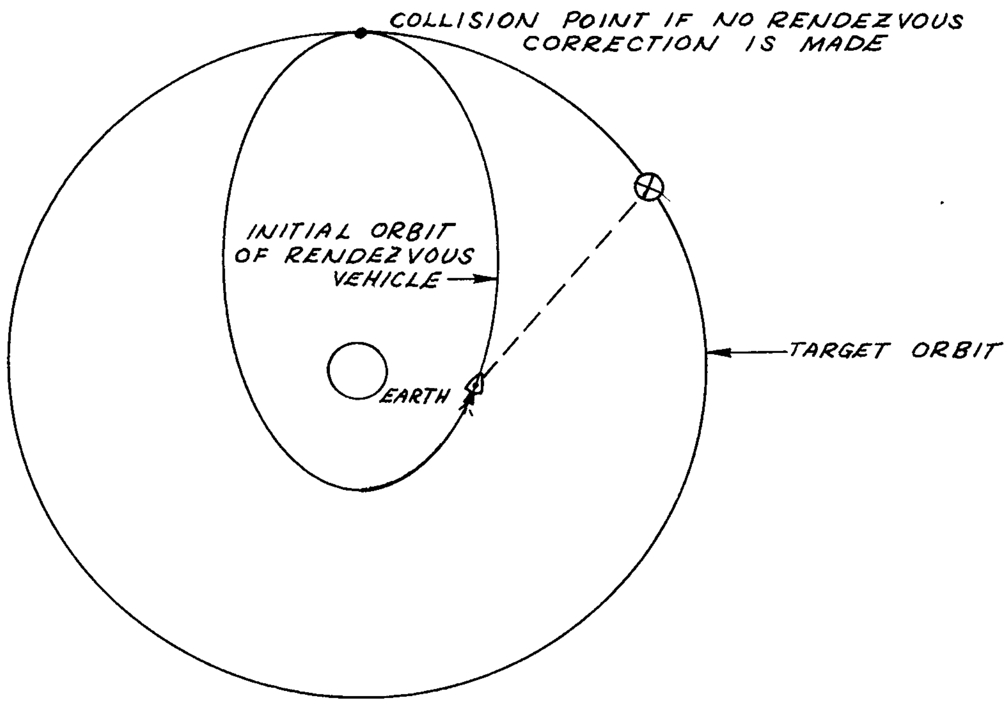


Figure 3.7 Example Rendezvous Orbits

2.3.1.4.3 Application of the Pontryagin Maximum Principle. This section is intended to illustrate the equivalence of Dynamic Programming and the Pontryagin Maximum Principle by determining the control function given in Equation (3.51) from an application of the Maximum Principle. As before, the cost function J, and the system equation are

$$J = x_f^T x_f + \lambda \int_{t_0}^{t_f} u^T u \, d\tau \tag{3.52}$$

$$\dot{x} = Ax + Bu$$

Thus, the hamiltonian is

$$H = -\lambda u^T u + P^T (Ax + Bu)$$

and maximizing H with respect to u gives

$$u = \frac{1}{2} \frac{B^T P}{\lambda} \tag{3.53}$$

At this point, the co-state equations are obtained from the hamiltonian as

$$\dot{P} = -\frac{\partial H}{\partial x}$$

or

$$\dot{P} = -A^T x \tag{3.54}$$

Now, since there are no terminal constraints on the problem, the boundary conditions on P are given by

$$P(t_f) = -2\, x(t_f) \tag{3.55}$$

For convenience, the equations defining the state vector, x, and the co-state vector, P, will be combined as

$$\begin{pmatrix} \dot{x} \\ \hline \dot{P} \end{pmatrix} = \left(\begin{array}{c|c} A & \frac{BB^T}{\lambda} \\ \hline 0 & -A \end{array}\right) \begin{pmatrix} x \\ \hline P \end{pmatrix}$$

This equation is a linear, homogenous, first order differential equation. Thus, the transition matrix (solution) for this equation, denoted by $\psi(t,t_0)$, is

$$\begin{pmatrix} x(t) \\ \hline p(t) \end{pmatrix} = \psi(t, t_0) \begin{pmatrix} x(t_0) \\ \hline p(t_0) \end{pmatrix}$$

Now, if t is replaced by t_f, and t_0 by t and if the transition matrix is partitioned, the following equations can be written

$$x(t_f) = \psi_{11} x(t) + \psi_{12} p(t)$$

$$p(t_f) = \psi_{12} x(t) + \psi_{22} p(t)$$

Thus, the boundary conditions (3.55) can be used to express $p(t_f)$ in terms of $x(t_f)$ as

$$p(t) = -(\psi_{12} + \tfrac{1}{2}\psi_{22})^{-1}(\psi_{11} + \tfrac{1}{2}\psi_{22})\, x(t)$$

or, defining the matrix $K(t)$ as

$$K(t) = \tfrac{1}{2}(\psi_{12} + \tfrac{1}{2}\psi_{22})^{-1}(\psi_{11} + \tfrac{1}{2}\psi_{22})$$

there results

$$p(t) = -2K(t)\, x(t) \tag{3.56}$$

and the control is

$$u = -\frac{B^T}{\lambda} K(t)\, x(t) \tag{3.57}$$

If Equation (3.56) is now differentiated, if (3.52) is substituted for $\dot{x}$, and if (3.57) is substituted for u, the result is

$$\dot{P}(t) = -\left[\dot{K} + KA - \frac{BB^T}{\lambda} K\right] x \qquad (3.58)$$

But, if (3.56) is substituted into (3.54)

$$\dot{P}(t) = A^T K(t) x(t) \qquad (3.59)$$

Thus, it is apparent that these two expressions can be equivalent only if $K(t)$ obeys the equation

$$\dot{K}(t) = -A^T K - KA + \frac{BB^T}{\lambda} K \qquad (3.60)$$

Finally, it is noted that Equation (3.60) is identical to the last equation of (3.44); thus, as before, the solution for $K(t)$ is given in Equation (3.50). Substituting this solution in the expression for the control (3.57) gives

$$u = -B^T e^{A^T(t_f - t)} \left[\lambda I + H(t_f - t)\right]^{-1} e^{A(t_f - t)} x(t)$$

This control is identical to that found in the last section using Dynamic Programming.

2.3.1.5 Cooperative Rendezvous

In this section a rendezvous problem is considered in which two vehicles, initially in different orbits, are both maneuvered with the result that the two vehicles achieve a rendezvous while minimizing some performance criteria. Notice that since both vehicles are maneuvering the final orbit which results when rendezvous has occurred is not necessarily the same as the initial orbit of either vehicle. The discussion of the cooperative rendezvous problem presented below is a review of Reference 3.7.

2.3.1.5.1 <u>State Equations and Optimization Criteria</u>. Since both vehicles have the ability to apply thrust and thus alter their orbits, the equations developed in Section 2.3.1.1 which relate the motion of one vehicle to the other will require modification. This modification is easily accomplished, however, by assuming an intermediate reference orbit which contains a ficticious vehicle coming the origin of the coordinate system heretofore carried by the target vehicle. Notice that this representation presents a potential mechanization problem, in that a direct measurement of position and velocity relative to the imaginary point in space is not possible (only position relative to the other spacecraft can be observed); rather recourse must be made to the absolute motion of the vehicle relative to the center of attraction and to the motion of the ficticious vehicle in the reference.

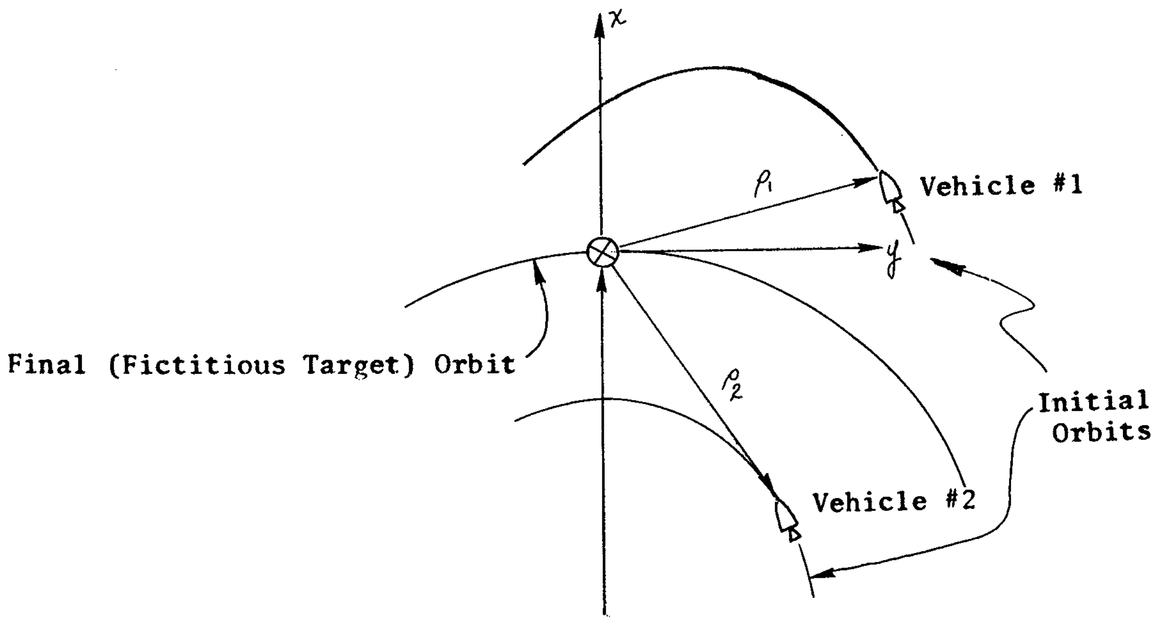


Figure 3.8 Cooperative Rendezvous

As discussed earlier, the motion of the first vehicle relative to the ficticious vehicle in the reference orbit is governed by the equation

$$\dot{\underline{x}}^{(1)} = A\underline{x}^{(1)} + f^{(1)}(u^{(1)})$$

and that of the second vehicle (see Figure 3.8) by

$$\dot{\underline{x}}^{(2)} = A\underline{x}^{(2)} + f(u^{(2)})$$

where the vector $\underline{x}^{(1)}$, the matrix A, and the forcing function $f(u^{(1)})$ are as defined in Section 2.3.1.1. A rendezvous occurs when

$$\underline{x}^{(1)}(t) = \underline{x}^{(2)}(t)$$

This rendezvous is to be accomplished by minimizing a function of the states and controls. For this discussion, only a general cost function of the form

$$J = \int_0^{t_f} f^o(\underline{x}^{(1)}, \underline{x}^{(2)}, u^{(1)}, u^{(2)})\,dt$$

is considered. This cost function can be rewritten in terms of a new variable $x^{(0)}$ as

$$\dot{x}^{(0)} = f^o(\underline{x}^{(1)}, \underline{x}^{(2)}, u^{(1)}, u^{(2)})$$

$$x^{(0)}(0) = 0, \quad x^o(t_f) = J$$

The problem can now be reformulated by defining a new state vector $\underline{x}$, a new forcing function $\underline{f}$, and a new control vector $\underline{u}$ as

$$\underline{x} = \begin{bmatrix} x^{(0)} \\ \underline{x}^{(1)} \\ \underline{x}^{(2)} \end{bmatrix} \qquad f = \begin{bmatrix} f^{(0)} \\ \underline{f}(u^{(1)}) \\ \underline{f}(u^{(2)}) \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} \underline{u}^{(1)} \\ \underline{u}^{(2)} \end{bmatrix}$$

the state equation becomes

$$\dot{\underline{x}} = \underset{\sim}{A}\underline{x} + \underline{f}(\underline{u}) \qquad (3.61)$$

where

$$\underset{\sim}{A} \triangleq \left[\begin{array}{c|c|c} O & O & O \\ \hline O & A & O \\ \hline O & O & A \end{array}\right] \qquad (3.62)$$

2.3.1.5.2 <u>Optimization by the Pontryagin Maximum Principle.</u> Application of the maximum principle to this problem results in equations similar to those obtained in previous sections due to the similarity of Equation (3.61) to the state equations previously used. It is the nature of the boundary conditions which distinguishes this problem from those discussed earlier. This fact can be seen in the following discussion.

$$H = p^{(o)}x^{(o)} + p^{T}\left[\underset{\sim}{A}x + F(u)\right] = p^{(1)^T}\left[Ax^{(1)} + F(u^{(1)})\right] + p^{(2)^T}\left[Ax^{(2)} + F(u^{(2)})\right] + p^{(o)}x^{(o)}$$

So that the co-state equations, given by,

$$\dot{p} = -\frac{\partial H}{\partial x}$$

are similar in form (except for the p^{o} co-state), due to the uncoupled nature of the system matrix $\underset{\sim}{A}$, to those obtained previously, i.e.,

$$\dot{p} = -\underset{\sim}{A}^{T}p \qquad (3.63)$$

or

$$\dot{p}^{(o)} = -p^{(o)}$$

$$\dot{p}^{(1)} = -A^{T}p^{(1)}$$

$$\dot{p}^{(2)} = -A^{T}p^{(2)}$$

But, as before for this type of problem, the optimum control, $\underset{\sim}{u}$, leads to a set of equations similar to (3.20) and (3.21), i.e.,

$$\tan\theta_1 = \frac{p_4^{(1)}}{p_3^{(1)}} \qquad \tan\theta_2 = \frac{p_4^{(2)}}{p_3^{(2)}}$$

and

$$a_1 = \begin{cases} a_{MAX}^{(1)} & (P_3^{(1)} \cos\theta_1 + P_4^{(1)} \sin\theta_1) > 0 \\ 0 & (P_3^{(1)} \cos\theta_1 + P_4^{(1)} \sin\theta_1) < 0 \end{cases}$$

$$a_2 = \begin{cases} a_{MAX}^{(2)} & (P_3^{(2)} \cos\theta_2 + P_4^{(2)} \sin\theta_2) > 0 \\ 0 & (P_3^{(2)} \cos\theta_2 + P_4^{(2)} \sin\theta_2) < 0 \end{cases}$$

where θ is the direction of the thrust and $a^{(1)}$ is the thrust magnitude (see Section 2.3.1.1 for clarification of these variables).

If the individual state vectors ($\underline{X}^{(1)}$ and $\underline{X}^{(2)}$) are n dimensional, then Equations (3.61) and (3.63) are a set of $4n+2$ first order differential equations; therefore, $2n+2$ boundary conditions are required before a solution can be generated. The initial state of the system will be known; thus, $n+1$ initial conditions will be

$$x^{(0)}(t_0) = 0 \tag{3.64}$$

$$x^{(i)}(t_0) = 0 \qquad i = 1, n$$

Further, since there are no terminal constraints on the $X^{(0)}$ component a terminal boundary condition for the co-state component $P^{(0)}(t_f)$ is

$$p^{(0)}(t_f) + \cancelto{0}{\frac{\partial \psi}{\partial x^{(0)}}} + \frac{\partial \phi}{\partial x^{(0)}}(x_f, t_f) = 0$$

$$\therefore \; p^{(0)}(t_f) = -1 \tag{3.65}$$

Now there are n terminal constraints on the state vector, i. e.,.

$$\begin{aligned}
\psi_1(\underline{x}_f) &= x^{(1)}(t_f) - x^{(n+1)}(t_f) = 0 \\
\psi_2(\underline{x}_f) &= x^{(2)}(t_f) - x^{(n+2)}(t_f) = 0 \\
&\vdots \\
\psi_n(\underline{x}_f) &= x^{(n)}(t_f) - x^{(2n)}(t_f) = 0
\end{aligned} \tag{3.66}$$

and therefore there are n terminal constraints on the co-state vector obtained from the relation

$$p^{(i)}(t_f) = -\sum_j \mu_j \left. \frac{\partial \psi_j}{\partial x_i} \right|_{t=t_f}$$

Thus, for example, the first and $(n+1)^{th}$ co-state variables have boundary conditions

$$p^{(1)}(t_f) = -\mu_1 \frac{\partial (x^{(1)} - x^{(n+1)})}{\partial x^{(1)}} = -\mu_1$$

$$p^{(n+1)}(t_f) = -\mu_1 \frac{\partial (x^{(1)} - x^{(n+1)})}{\partial x^{(n+1)}} = \mu_1$$

or

$$p^{(1)}(t_f) = -p^{(n+1)}(t_f)$$

A similar expression exists for the remaining components resulting in the set

$$\begin{aligned}
p^{(1)}(t_f) - p^{(n+1)}(t_f) &= 0 \\
p^{(2)}(t_f) - p^{(n+2)}(t_f) &= 0 \\
&\vdots \\
p^{(n)}(t_f) - p^{(2n)}(t_f) &= 0
\end{aligned} \tag{3.67}$$

Thus, the $2n$ equations comprising (3.66) and (3.67) specify $2n$ boundary conditions on the $4n$ unknown variables, $X^{(i)}(t_f)$, $P^{(i)}(t_f)$ $i=1, 2n$ in terms of the remaining $2n$ variables. Therefore, Equations (3.64), (3.65), and (3.66) specify the required number of boundary conditions.

At this point a numerical solution can be generated using one of the techniques described in Reference (3.5).

2.3.2 Orbital Transfer Problems

The problem considered in this section is that of determining the trajectory along which a vehicle can be transferred from one orbit to another while minimizing some performance index, usually time or fuel. The rendezvous problem discussed in Section 2.3.1 can be considered to be a special case of the orbital transfer problem in which an additional constraint requiring insertion at a particular point and a specified time is given and for which the motion of all masses can be described by linear equations. For the previous case, these assumptions led to very simple boundary conditions, i.e., $\underline{X}(t_f) = 0$. In the present problem, however, while a two-body orbit can be defined in terms of six constants (known as orbital elements) or any six independent quantities which are known at some time (for example, the components of position and velocity), the orbit must be expressed by equations relating the orbital elements to the quantities of interest; the result is that the boundary conditions become time varying quantities. One way to avoid the time varying boundary conditions would be to choose the elements of the state vector as orbital elements; however, in this case, the description of the thrust components is generally quite involved. Another technique is to use some other quantity which can describe both the orbit and the control without a great deal of difficulty. An example of the latter approach is given in Section 2.3.2.1.2 where polar coordinates are used for position and angular momentum (radial rate completes the description).

Three types of propulsion systems are considered; (1) thrust limited, (2) power limited, and (3) impulsive systems. In these discussions, the general non-linear problems are formulated and, for the thrust limited and power limited systems, a linear formulation is developed.

2.3.2.1 Thrust Limited Vehicles

Chemical rocket motors are members of a class of thrust limited vehicles, i.e., the maximum thrust available is bounded. The thrust of such a device has been developed in Section 2.3.1.3.1 and is reproduced below.

$$T = \frac{c}{m} \dot{m}$$

where $\dot{m}$ = mass flow rate
c = rocket exhaust velocity
m = vehicle mass

Generally for these systems, the exhaust velocity is constant, and the thrust level is controlled by varying the mass flow rate.

2.3.2.1.1 Formulation of the State and Optimization Equations. The equations of motion will be expressed in an inertial coordinate system whose origin is at the center of the central force field. The state vector will be composed of seven elements, those of which are position coordinates, three of which are velocity coordinates, and the seventh is the mass of the vehicle

$$\begin{array}{lll} x_1 \triangleq x & x_4 \triangleq \dot{x} & x_7 \triangleq m \\ x_2 \triangleq y & x_5 \triangleq \dot{y} & \\ x_3 \triangleq z & x_6 \triangleq \dot{z} & \end{array}$$

Thus, the state equations are

$$\begin{array}{lll} \dot{x}_1 = x_4 & \dot{x}_4 = \dfrac{c}{x_7} \beta \ell_1 - g_1 & \dot{x}_7 = -\beta \\ \dot{x}_2 = x_5 & \dot{x}_5 = \dfrac{c}{x_7} \beta \ell_2 - g_2 & \\ \dot{x}_3 = x_6 & \dot{x}_6 = \dfrac{c}{x_7} \beta \ell_3 - g_3 & \end{array} \tag{3.68}$$

where β is the mass flow rate, where the ℓ_i s are direction cosines between the coordinate axis and the rocket thrust vector (the control variables are ℓ_i and β), and where the gravitational acceleration components have the form

$$g_i = \frac{-\mu x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}$$

(μ is the gravitational constant for the force field).

The function to be minimized in this maneuver is the characteristic velocity, ΔV_C , which is defined in terms of the initial and final mass as

$$\Delta V_c = C \ln (m_o / m_f) \tag{3.69}$$

Thus, the hamiltonian is

$$H = p_1 x_4 + p_2 x_5 + p_3 x_6 + p_4 \left(\frac{C}{x_7} \beta l_1 + g_1 \right) + p_5 \left(\frac{C}{x_7} \beta l_2 + g_2 \right)$$

$$+ p_6 \left(\frac{C}{x_7} \beta l_3 + g_3 \right) + p_7 (-\beta) \tag{3.70}$$

and the co-state equations (from the relation)

$$\dot{p}_i = - \frac{\partial H}{\partial x_i}$$

are

$$\begin{aligned}
\dot{p}_1 &= - \left(p_4 \frac{\partial g_1}{\partial x_1} + p_5 \frac{\partial g_2}{\partial x_1} + p_6 \frac{\partial g_3}{\partial x_1} \right) \\
\dot{p}_2 &= - \left(p_4 \frac{\partial g_1}{\partial x_2} + p_5 \frac{\partial g_2}{\partial x_2} + p_6 \frac{\partial g_3}{\partial x_2} \right) \\
\dot{p}_3 &= - \left(p_4 \frac{\partial g_1}{\partial x_3} + p_5 \frac{\partial g_2}{\partial x_3} + p_6 \frac{\partial g_3}{\partial x_3} \right) \\
\dot{p}_4 &= - p_1 \\
\dot{p}_5 &= - p_2 \\
\dot{p}_6 &= - p_3 \\
\dot{p}_7 &= \frac{C}{x_7^2} \beta (p_4 l_1 + p_5 l_2 + p_6 l_3)
\end{aligned} \tag{3.71}$$

Since the hamiltonian is linear in the control variable, it is obvious that if no constraint is placed on the magnitude of β the hamiltonian will be maximized by $\beta = \pm\infty$. However, since β represents the mass flow rate, it must be constrained by the equation

$$0 \leq \beta \leq \beta_{MAX} \tag{3.72}$$

As a point of interest, it is noted that it may be desirable to insist that the rocket motor be non-throttleable since this is a simpler and more reliable physical situation, and since most liquid motors have this characteristic to a large degree. In this case, the mass flow rate should be constrained to be either zero or β_{MAX} . A mathematical expression of this type of constraint requires that β obey the equation

$$\beta(\beta - \beta_{MAX}) = 0$$

However, for the hamiltonian of Equation (3.70) no intermediate values of β are necessary for an extremal with the result that the constraint Equation (3.72) and that given above yield the same answer.

In addition to the constraint on the magnitude of the flow rate there is a contraint on the direction cosines, i.e.,

$$\ell_1^2 + \ell_2^2 + \ell_3^2 = 1 \tag{3.73}$$

There are several ways to handle this constraint such as defining a new variable, rewriting the direction cosines in terms of two angles of a spherical coördinate system, or using the method of Lagrange multipliers. The latter method will be illustrated. Let $\underset{\sim}{H}$ be the augmented hamiltonian, then if only the terms in the original hamiltonian which contain the control variables are retained

$$\underset{\sim}{H} = \frac{c}{x_7}\beta(P_4\ell_1 + P_5\ell_2 + P_6\ell_3) - P_7\beta + \lambda(\ell_1^2 + \ell_2^2 + \ell_3^2 - 1)$$

Maximizing $\underset{\sim}{H}$ with respect to ℓ_1, ℓ_2, ℓ_3 results in the equations

$$\frac{c\beta}{x_7} P_4 + 2\lambda \ell_1 = 0$$

$$\frac{c\beta}{x_7} P_5 + 2\lambda \ell_2 = 0 \qquad (3.74)$$

$$\frac{c\beta}{x_7} P_6 + 2\lambda \ell_3 = 0$$

This set along with the constraing Equation (3.73) can be solved for the four variables ℓ_1, ℓ_2, ℓ_3, and λ. Maximizing H with respect to the variable β gives

$$\beta = \begin{cases} 0 & (P_4\ell_1 + P_5\ell_2 + P_6\ell_3 - P_7) < 0 \\ \beta_{MAX} & (P_4\ell_1 + P_5\ell_2 + P_6\ell_3 - P_7) > 0 \end{cases} \qquad (3.75)$$

Thus, the optimum control is obtained from Equations (3.73), (3.74), and (3.75) in terms of the co-state variables P. Because of the non-linear nature of the state and co-state equations, only numerical solutions are possible. Some insight to the problem can, however, be gained by some further manipulation of the co-state equations as suggested by Lawden (Reference 3.8). The fourth, fifth, and sixth co-state variables will be considered as a three vector called, by Lawden, the "primer" vector. From Equation (3.74) it is seen that this vector is parallel to the thrust vector, a result identical to that obtained in the two dimensional linear rendezvous problem. Further, the set of Equations (3.74) can be inverted to obtain the direction cosines, ℓ_i, in terms of the components of the primer.

$$\ell_1 = P_4 \Big/ \sqrt{P_4^2 + P_5^2 + P_6^2}$$

$$\ell_2 = P_5 \Big/ \sqrt{P_4^2 + P_5^2 + P_6^2} \qquad (3.76)$$

$$\ell_3 = P_6 \Big/ \sqrt{P_4^2 + P_5^2 + P_6^2}$$

Alternately, note that for periods of maximum thrust Equation (3. 75) indicates that this quantity is positive, and for periods of zero thrust the same quantity is negative. Therefore, at the transition point from maximum thrust to zero thrust, or vice versa, this quantity is zero. This set of conditions must be satisfied whether or not the hamiltonian is zero, i. e., whether or not the final time is specified. Making use of Equation (3. 76) this relation becomes

$$P_7 = \frac{c}{x_7}\sqrt{P_4^2 + P_5^2 + P_6^2} \tag{3.78}$$

with a boundary condition at the terminal orbit obtained from the relation:

$$P_7(t) + \frac{\partial \phi(\underline{x}_f)}{\partial x_7} = 0$$

The cost function $\phi(\underline{x}_f)$ is given by Equation (3. 69), and in terms of the state variables it is

$$\phi(\underline{x}_f) = c \ln(m_o/x_7)$$

Therefore, the boundary condition becomes

$$P_7(t_f) = \frac{c}{x_7(t_f)} \tag{3.79}$$

As a final topic in this section, a solution for the primer vector along a coasting arc will be obtained in terms of six constants of integration. First, note that the first three equations of (4. 71) can be combined with the second three to form a set of second order differential equations for the primer vector. If $\underset{\sim}{P}$ is defined as the primer vector, i. e.

$$\underset{\sim}{P} = \begin{bmatrix} P_4 \\ P_5 \\ P_6 \end{bmatrix}$$

then this second order differential equation can be written as

$$\ddot{\underset{\sim}{P}} = \left[\frac{\partial \underline{g}}{\partial \underline{x}}\right] \underset{\sim}{P}$$

Substitution of these expressions back in Equation (3.74) results in an expression for the unknown multiplier λ.

$$\lambda = \frac{c\beta}{2x_7}\sqrt{P_4^2 + P_5^2 + P_6^2}$$

Now, if Equation (3.8) is substituted in the differential equation for $\dot{P}_7$ it is seen that

$$\dot{P}_7 = \frac{c\beta}{x_7^2}\sqrt{P_1^2 + P_2^2 + P_3^2} \tag{3.77}$$

But, the hamiltonian of Equation (3.70) is independent of time; thus, since the terminal time is not specified its value is a constant equal to zero at all points on the optimal trajectory. Consider a coasting arc, i.e., one where $\beta = 0$. Since the hamiltonian must be zero at all points on the trajectory, it follows that all the terms in the hamiltonian which do not contain β as a factor must cancel each other. On the other hand, if β is non-zero the hamiltonian is still required to be zero and since the terms which do not contain β as a factor are zero it follows (by considering the situation an instant after the rocket motor has been turned on) from the continuity of the co-state variables, that the factors which contain β must be zero. Thus, at points on the trajectory where the rocket is turned on or off the following condition must hold:

$$\frac{c}{x_1}\beta(P_4 \ell_1 + P_5 \ell_2 + P_6 \ell_3) - \frac{P_7}{\beta} = 0$$

where

$$\beta \neq 0$$

The matrix in this equation is obtained from (4.71). Now, if the reference coordinate system is shifted (call the coordinate vectors of this system λ, σ, ν) so that its origin is at a distance r from the attracting body (where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

and rotating with an angular speed $\dot{\theta}$ such that the coordinate λ remains aligned with the radius vector, (see sketch)

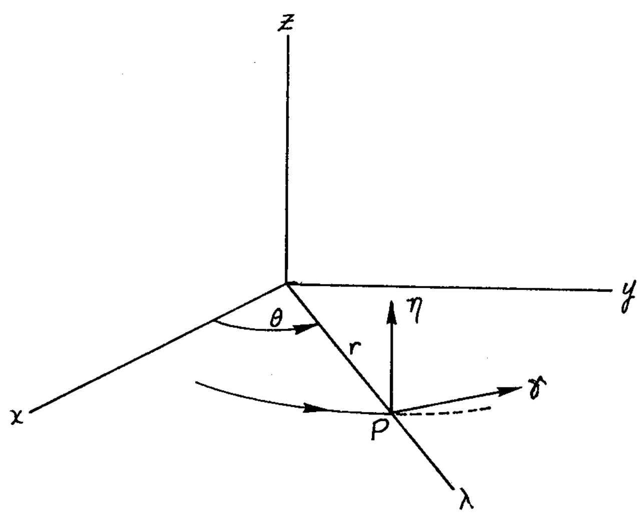


Figure 3.9 Coordinate Reference

this matrix becomes

$$\left[\frac{\partial q}{\partial x}\right] = \begin{bmatrix} \frac{\partial \mu}{r^2} & 0 & 0 \\ 0 & \frac{-\mu}{r^3} & 0 \\ 0 & 0 & \frac{-\mu}{r^2} \end{bmatrix}$$

Note that the zeros appear because the choice of the coordinate system requires $\eta = \gamma = 0$; thus, the primer vector is now expressed in the rotating system. However, in the rotating coordinate system, the second derivative of the primer vector must be modified by the addition of coriolis and centripital terms, as

$$\ddot{\underset{\sim}{p}}_r = \ddot{\underset{\sim}{p}} + 2\dot{\theta}\,\hat{\gamma} + \dot{\underset{\sim}{p}} + \ddot{\theta}\,\gamma + \underset{\sim}{p} + \dot{\theta}^2(\hat{\gamma}\cdot\underset{\sim}{p}\,\hat{\gamma} - \underset{\sim}{p})$$

where $\hat{\gamma}$ is a unit vector along the γ axis. Making a change of variables defined by

$$\underset{\sim}{q} = r\underset{\sim}{p}\,, \quad \underset{\sim}{q} \equiv \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

and using the polar equations for motion in a central force field

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2}\,, \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

the equation of $\ddot{\underset{\sim}{p}}$ becomes

$$\ddot{\underset{\sim}{p}} = r\ddot{\underset{\sim}{q}} + 2\dot{r}\dot{\underset{\sim}{q}} + 2r\dot{\theta}\,\hat{\gamma} \times \underset{\sim}{q} + r\dot{\theta}^2\,\hat{\gamma}\cdot\underset{\sim}{q}\,\gamma - \frac{\mu}{r^2}\underset{\sim}{q}$$

But, this expression can be rewritten in terms of derivative with respect to θ (denoted by primes) as

$$\ddot{\underset{\sim}{p}} = r\dot{\theta}^2\left(\underset{\sim}{q}'' + 2\,\hat{\gamma} \times \underset{\sim}{q}' + \hat{\gamma}\cdot\underset{\sim}{q}\,\hat{\gamma} - \frac{\mu}{r^3\dot{\theta}^2}\right)$$

Equating this expression to $\left[\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}}\right]\underset{\sim}{p}$ and making use of the two body relation $r^2\dot{\theta} = \sqrt{\mu a}$ (where a is the semimajor axis of the two body orbit) gives the following set of equations.

$$
\begin{aligned}
u'' - 2v' - \frac{3r}{a}u &= 0 \\
v'' + 2u' &= 0 \\
\omega'' + \omega &= 0
\end{aligned}
\tag{3.80}
$$

If the orbit is circular, then $a = r$ and a solution to this set (in terms of the components of the primer in the rotating system) is

$$
\begin{aligned}
P_4 &= A \cos(\theta - \omega) + B \sin(\theta - \omega) + 2C \\
P_5 &= 2B \cos(\theta - \omega) - 2A \sin(\theta - \omega) - 3C(\theta - \omega) + D \\
P_6 &= E \cos(\theta - \omega) + F \sin \omega
\end{aligned}
\tag{3.81}
$$

where ω is an arbitrary reference direction (equal to the argument of perigee for elliptical orbits). For elliptical orbits, the first equation is replaced by a "first integral" obtained by noting that the hamiltonian is a constant. In terms of the vector q, the hamiltonian is

$$
r\dot{r}\dot{\theta}u' + r^2\dot{\theta}^2 v' + (\dot{r}^2 + r^2\dot{\theta}^2 + \frac{\mu}{r})u = C \tag{3.82}
$$

Now, making use of two more expressions from two body orbit theory

$$
\begin{aligned}
\frac{a}{r} &= 1 + e \cos(\theta - \omega) \\
\dot{r} &= e\sqrt{\frac{\mu}{a}} \sin(\theta - \omega)
\end{aligned}
\tag{3.83}
$$

and integrating the second equation of (3.80) gives

$$
v' = Ae - 2u \tag{3.84}
$$

So that combining Equations (3.82) and (3.84) enables a solution for U to be obtained

$$
u = (1 + e\cos(\theta - \omega))(A\cos(\theta - \omega) + Bc\sin(\theta - \omega) + CI) \tag{3.85}
$$

where

$$I = \sin f \int \frac{df}{\sin^2 f(1+e\cos f)^2}$$

where

$$f = \theta - \omega$$

(the reader is directed to reference 3.9 for an evaluation of the integral for I in terms of elementary functions). Thus, substitution of (3.85) into (3.84) and integrating yields:

$$V = \left[1+e\cos(\theta-\omega)\right]\left\{-A\sin(\theta-\omega)+B\left[1+e\cos(\theta-\omega)\right]\right.$$
$$\left.+\frac{D-A\sin(\theta-\omega)}{1+e\cos(\theta-\omega)}+CJ\right\}$$

where

$$J = \frac{\cot(\theta-\omega)}{e(1+e\cos(\theta-\omega))} + \frac{1+e\cos(\theta-\omega)}{e\sin(\theta-\omega)}\, I$$

Finally, the third equation of (3.80) gives

$$\omega = E\cos(\theta-\omega) + F\sin(\theta-\omega)$$

so that a complete solution for the primer vector for non-circular orbits can be written as

$$P_4 = A\cos(\theta-\omega) + Be\sin(\theta-\omega) + CI$$

$$P_5 = -A\sin(\theta-\omega) + B\left[1+e\cos(\theta-\omega)\right] + \frac{D-A\sin(\theta-\omega)}{1+e\cos(\theta-\omega)} + CJ \quad (3.86)$$

$$P_6 = \left[1+e\cos(\theta-\omega\right]^{-1}\left[E\cos(\theta-\omega)+F\sin(\theta-\omega)\right]$$

For the general case of finite length thrust intervals, the equations just presented represent the extent of the analytic formulation and solution for this problem. In order to study the finite transfer any further, a numerical solution to the two point boundary value problem which represents the solution to the state and co-state equations must be made. One method which has been applied to the orbit transfer problem by McGill and Kenneth

(Reference 3. 12) and by McCue (Reference 3. 11), is that of quasilinearization (Newton-Raphson iteration). This technique consists of linearizing the state equations so that the unknown boundary conditions can be obtained in terms of the known boundary conditions. The procedure is to first form a vector which consists of the state and co-state vector, i. e.,

$$y = \begin{pmatrix} x \\ \hline p \end{pmatrix}$$

Thus, if the state equations satisfy the equation

$$\dot{x} = f_1(x, p, u)$$

and co-state equations satisfy the equation

$$\dot{p} = f_2(x, p, u)$$

(where $f_1(x, p, u)$ and $f_2(x, p, u)$ are in this case given by Equations (3. 68) and (3. 71)), the combined vector obeys the equation

$$\dot{y} = \begin{pmatrix} f_1(x, p, u) \\ \hline f_2(x, p, u) \end{pmatrix} = F(Y, u) \tag{3.87}$$

However, instead of requiring that Y obey this equation, it will only be required that Y obey a linearized form of that equation where $F(y, u)$ has been expanded in a Taylor series (truncated after the linear term and evaluated about some nominal Y). For example, call the nominal Y value Y_n then the equation becomes

$$\dot{y} = F\left[Y_n, u(Y_n)\right] + \frac{\partial F}{\partial Y_n}\left[Y_n, u(Y_n)\right]\left[Y - Y_n\right] \tag{3.88}$$

Notice that the control has been written as a function of the nominal state vector. Since this equation is only a linear approximation of true equation, an iteration will be performed with Y_n replaced by Y for the next iteration. That is, an iteration sequence is defined by

$$\dot{Y}_{n+1} = F(Y_n) + \frac{\partial F(Y_n)}{\partial Y_n}(Y_{n+1} - Y_n) \tag{3.89}$$

This equation implies that an initial state vector, $Y_0(t)$, must be somehow obtained. It may be possible to obtain this starting function on the basis of some physical reasoning or it may be a pure guess. The only requirement is that Y_0 satisfy the known boundary conditions. In Reference (3. 10), McGill and Kenneth prove that if the initial choice is such that the sequence (3. 89) converges, the function to which it converges is unique and is the solution to (3. 87).

Equation (3. 87) cannot be solved directly because of the mixed nature of the boundary conditions. However, Equation (3. 89) is linear in the unknown vector Y_{n+1} and can be solved directly as follows. The first step in the solution is to rewrite Equation (3. 89) in a more familiar form

$$\dot{Y}_{n+1} = G(Y_n)Y_{n+1} + g(Y_n) \tag{3.90}$$

where $$G(Y_n) = \frac{\partial F(Y_n)}{\partial Y_n}, \qquad g(Y_n) = F(Y_n) + \frac{\partial F(Y_n)Y_n}{\partial Y_n}$$

Now, a solution to Equation (3. 90) can be written in terms of its transition matrix, $\phi(t_0, T)$

$$Y_{n+1}(T) = \left(\begin{array}{c|c} \phi_{11} & \phi_{12} \\ \hline \phi_{21} & \phi_{22} \end{array}\right) Y_{n+1}(t_0) + \int_{t_0}^{T} \phi(T, \tau)\, g(Y_n)\, d\tau \tag{3.91}$$

Notice that the integration can be readily performed (numerically if necessary) since the integrand is defined as a function of τ. Suppose that initial conditions on the X vector are known, and that terminal conditions on P are known (this choice is made only for convenience of illustration and the theory). From Equation (3. 91) the terminal values of P are obtained in terms of the known initial values of X, the unknown initial P values, and the appropriate components of the integral (call these I_P)

$$P(T) = \phi_{11} X(0) + \phi_{12} P(0) + I_p$$

Then $P(0)$ can be obtained as

$$P(0) = \phi_{12}^{-1} P(T) - I_p - \phi_{11} X(0)$$

Now the entire initial vector Y_{n+1} is known and $Y_{n+1}(t)$ can be obtained from Equation (3. 91) with T replaced by t. Note that the complete transition matrix $\phi(t_o, T)$ need not be known; since this is the case, if $\varphi(t, T)$ cannot be obtained analytically, only the portion necessary for the solution needed can be obtained by noting that the columns of $\varphi(t_o, t)$ represent independent solutions corresponding to Equation (3. 90) with a unit initial condition on one variable and zero on the rest. Thus, a numerical solution for the first column of $\phi(t_o, t)$ for example, can be obtained by numerically integrating

$$\dot{y}_{n+1} = G(y_n)\, y_{n+1}$$

with

$$y_{n+1}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

McCue, Reference (3. 11), applied this technique to a minimum fuel problem similar to that developed in this section. His results indicate that quasilinearization can be a powerful tool for this problem but that a good first approximation is required to insure convergence; a good first approximation was found by generating the corresponding impulsive transfer maneuver. Kenneth and McGill (Reference 3. 12) applied quasilinearation to a minimum time orbit transfer problem. However, since the theory presented requires that the final time be known, a slight modification is required. To illustrate this modification, assume that a two dimensional problem is being considered in which the state variables are radius (r), radial velocity (V_r) , and velocity normal to the radial direction The boundary conditions on the transfer are given by

$$\begin{array}{ll}
t_o = 0 & t_f = \\
r(t_o) = r_o & r(t_f) = r_f \\
V_r(t_o) = V_{ro} & V(t_f) = V_{rf} \\
V_n(t_o) = V_{no} & V_n(t_f) = V_{nf}
\end{array}$$

If a final time is guessed, then the boundary conditions are sufficient to determine a solution to the state and co-state equations. This solution, however, would not be optimum (unless the correct time is chosen) and the hamiltonian would not have the correct value. One means of adjusting the solution would be to use a one dimensional Newton-Raphson iteration on the final time to obtain the correct value for the hamiltonian; an alternative to performing the iteration on the hamiltonian (and the technique actually used by McGill and Kenneth) is to allow one of the terminal boundary conditions (say radius, since it should be monotone increasing with time) to be unsatisfied. That is, the two point boundary value problem is solved for the state and co-state vectors subject to the following conditions

$$t_o = 0 \qquad\qquad t_f = t_{f_o}$$

$$r(t_o) = r_o$$

$$V_r(t_o) = V_{ro} \qquad\qquad V_r(t_f) = V_{rf}$$

$$V_n(t_o) = V_{no} \qquad\qquad V_n(t_f) = V_{nf}$$

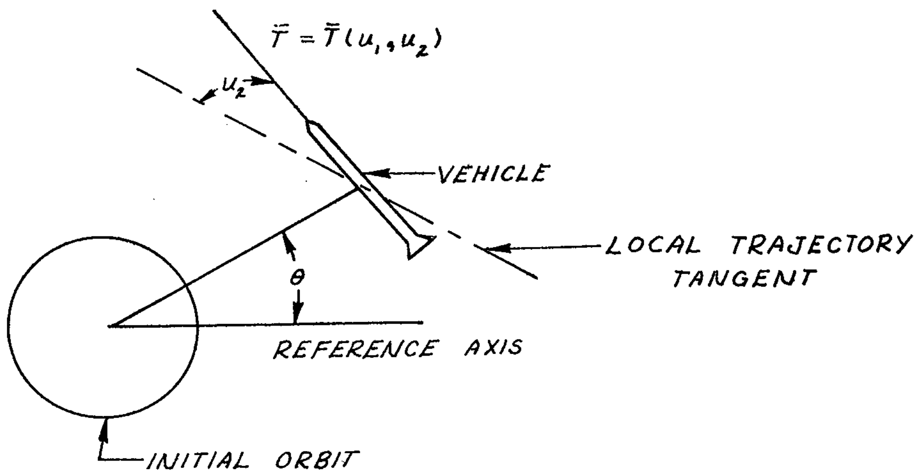
$$P_1(t_o) = 1$$

In this set, the final value of the radius has been dropped and the initial value of one of the co-state vectors added so that the total number of boundary conditions remain the same. The choice of the value one, for the initial co-state variable does not impose a great restriction on the problem since examination of the co-state equations and the expressions containing the co-state variables will reveal that an arbitrary scaling parameter can be applied to the co-state vector without changing the problem. The only trouble would arise if the correct value of $P_1(t_o)$ were zero. If this set of boundary conditions is used in the solution of the two point boundary value problem a final value of radius will be determined, however, if the guessed final time is not the correct one for the problem this value of range will not correspond to the required value of final radius and a Newton-Raphson iteration on final time as a function of final range can be performed as

$$t_{f_{(M+1)}} = t_{f_{(M)}} + \frac{(t_{f_{(M)}} - t_{f_{(M-1)}})}{r_{f_{(M}} - r_{f_{(M-1)}}} (r_f - r_{f_M})$$

2.3.2.1.2 Transfer Between Neighboring Circular Orbits. In this section, the special case of a transfer between circular, co-planar orbits of very nearly the same radius will be considered. The material for this section, including examples and illustrations, is a review of the PhD thesis of J. E. McIntyre (Reference 3.16).

It is assumed that the ratio of the radius of the initial orbit to the final orbit is $(1+\epsilon)$, where ϵ is a small quantity, and a solution is sought in powers of ϵ. Since the solution is to be expressed in powers of ϵ, it is convenient to formulate the problem in terms the polar coordinate system illustrated.



Transfer Between Neighboring Circular Orbits

To simplify the equations and obtain the most general results, the initial orbital radius is taken as the unit of length, the initial orbital velocity as the unit of velocity, the initial orbital period as the unit of time, and the initial mass of the vehicle as the unit of mass. With these units the equations of motion are

$$\frac{dr}{dt} = V \qquad \frac{dh}{dt} = r\, u_1 \cos u_2$$

$$\frac{d\theta}{dt} = \frac{h}{r^2} \qquad \frac{dV}{dt} = \frac{u_1 \sin u_2}{m} + \frac{h^2}{r^3} - \frac{1}{r^2}$$

$$\frac{dm}{dt} = -\frac{u_1}{c}$$

where

r = distance from center of attraction

h = angular momentum

V - radial velocity

θ - central angle

U_1 = thrust per g of the initial orbit $(0 \leq u_1 \leq u_{MAX})$

U_2 = steering angle

C = exhaust velocity

It is convenient, for this problem, to change from time as the independent variable to the central angle θ as the independent variable. This choice of independent variable serves to bring out certain geometrical similarities between the transfer trajectories between thrust limited vehicles and impulsive vehicles (impulsive orbit transfer is discussed in Section 2.3.2.3). Assuming that $d\theta/dt$ is always positive, the independent variable can be changed from t to θ by the operation

$$\frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\theta}$$

With this transformation, the equations of motion become

$$\frac{dr}{d\theta} = \frac{r^2 V}{h} \qquad \frac{dV}{d\theta} = \frac{r^2}{hm} u_1 \sin u_2 + \frac{h}{r} - \frac{1}{h} \qquad (3.92)$$

$$\frac{dh}{d\theta} = \frac{r^3}{hm} u_1 \cos u_2 \qquad \frac{dm}{d\theta} = \frac{-r^2}{hc} u_1$$

There are now four state variables, r, V, h, m, and two control variables u_1 and u_2. Before proceeding to linearize the equations of motion, note that if the thrust level were chosen independent of the change in the radius, ϵ, the maneuver would tend, with decreasing ϵ, toward the impulsive maneuver and much information would be lost. If, on the contrary, the thrust level is taken as

$$u_{1MAX} = A\epsilon$$

the duration of thrust application does not vanish with ϵ, and interesting information can be obtained even in the limit as ϵ goes to zero. The variable, A, is referred to as the "thrust level coefficient."

where the prime denotes differentiation with respec to θ . Thus maximizing the hamiltonian with respect to the control variables gives

$$\frac{\partial H}{\partial u_0} = - P_2 A u_1 \sin u_2 + P_3 A u_1 \cos u_2 = 0$$

or

$$\sin u_2 = \frac{P_3}{\sqrt{P_2^2 + P_3^2}} \qquad \cos u_2 = \frac{P_2}{\sqrt{P_2^2 + P_3^2}}$$

Now, since the hamiltonian is linear in the control u_1, it is maximized by U_{MAX} or zero depending on whether the coefficient of u_1 in the hamiltonian is positive or negative.

$$u_1 = u_{MAX} \qquad s > 0$$

$$u_1 = 0 \qquad s < 0$$

where S is the switching function given by

$$s = \sqrt{P_2^2 + P_3^2} + \frac{P_4}{c}$$

Since the control is a non-linear function of the co-state variables, recourse to numerical techniques is necessary even for this linearized problem. Before a numerical method can be applied, however, one more boundary condition is required. The additional boundary condition can be obtained once an optimization criterion has been selected. Consider first a minimum angle transfer. For this case, if the final mass is not specified, a boundary condition on the fourth co-state variable is obtained from

$$P_4 (0) = \sum \mu_i \frac{\partial \psi_i}{\partial m} + \frac{\partial \phi}{\partial m} (x_f, \theta_f)$$

where the ψ_i's are the terminal constraints and $\phi(x_f, \theta_f) = \phi_f$ In this case, the switching will never be negative and the engine will operate at maximum thrust throughout the transfer. The co-state equations can be solved in terms of the initial conditions on P_1, P_2, P_3.

The state variables, and the control variable u_1, are to be expressed as a series in powers of ϵ whose coefficients will be determined so that the equations of motion and the boundary conditions are satisfied. Since only linear terms are of interest in this section, the state variables have the form

$$r = r_0 + r_1 \epsilon \qquad m = m_0 + m_1 \epsilon$$

$$h = h_0 + h_1 \epsilon \qquad u_2 = u_{20} + u_{21} \epsilon$$

$$V = V_0 + V_1 \epsilon \qquad u_1 = u_1$$

where the terms with zero subscripts represent the values on the initial orbit. Now, if these expansions are substituted in the equations of motion (3.92), if terms of order higher than one are dropped (note that terms like ϵh_1^2, are second order), and if the numerical values of the initial conditions (i. e., $r_0 = 1$, $h_0 = 1$, $V_0 = 0$, and $m_0 = 1$) are substituted, the following equations result

$$\frac{dr_1}{d\theta} = V_1 \qquad \frac{dV_1}{d\theta} = A u_1 \sin u_{20} + 2h_1 - r_1$$

$$\frac{dh_1}{d\theta} = A u_1 \cos u_{20} \qquad \frac{dm_1}{d\theta} = -\frac{A u_1}{c}$$

where, the boundary conditions for these linearized equations are

$$r_1(t_0) = h_1(t_0) = V_1(t_0) = m_1(t_0) = 0$$

$$r(t_f) = 1 + \epsilon, \quad h(t_f) = 1 + \frac{\epsilon}{2}, \quad V_1(t_f) = 0$$

Optimization of the problem can now proceed formally. First, the hamiltonian is

$$H = P_1 V_1 + P_2 A u_1 \cos u_2 + P_3 (A u_1 \sin u_2 + 2h_1 - r_1) + P_4 \left(-\frac{A u_1}{c} \right)$$

and the co-state equations are

$$P_1' = -\frac{\partial H}{\partial r_1} = P_3 \qquad P_3' = -\frac{\partial H}{\partial V_1} = -P_1$$

$$P_2' = -\frac{\partial H}{\partial h_1} = -2P_3 \qquad P_4' = -\frac{\partial H}{\partial m_1} = 0$$

$$P_1(\theta) = P_1(0) \cos\theta + P_3(0) \sin\theta$$

$$P_2(\theta) = -2P_1(0) + P_2(0) + 2P_1(0)$$

$$P_3(\theta) = P_3(0) \cos\theta - P_1(\theta) \sin\theta$$

$$P_4(\theta) = 0$$

A numerical technique such as Neustadt's method discussed in Section 2.3.1.3.2 can now be used to determine the numerical values of $P(0)$. Such a method was used in the reference with the results indicated below. Figure 3.10 shows a variation of the thrust level coefficient as a function of the range angles for the minimum angle transfer and Figure 3.11 shows the variation of the steering program with radial position, r, for various thrust level coefficients.

Note that for thrust levels for which $A < 3\sqrt{3}/8\pi$ the steering angle lies in either the first or fourth quadrant, while for $A > 3\sqrt{3}/8\pi$ the steering angle goes through all four quadrants. For $A = 3\sqrt{3}/8\pi$ the steering program is discontinuous.

The half darkened points in Figure 3.10 and the dotted curve in Figure 3.11 indicate the minimum angle transfer with the additional restriction that the steering angle must lie in either the first or fourth quadrant. It can be concluded from these plots that there is little to be gained by having a vehicle with four quadrant rather than two quadrant capability.

For values of A such that

$$A = 1/4\pi N, \quad N = 1, 2, 3 \ldots$$

the final range is given by

$$\theta_f = 2N\pi, \quad N = 1, 2, 3 \ldots$$

and the steering program is circumferential with u_2 equal to zero over the entire trajectory. It is interesting to note in Figure 3.11 that the lower three values of A are chosen so that

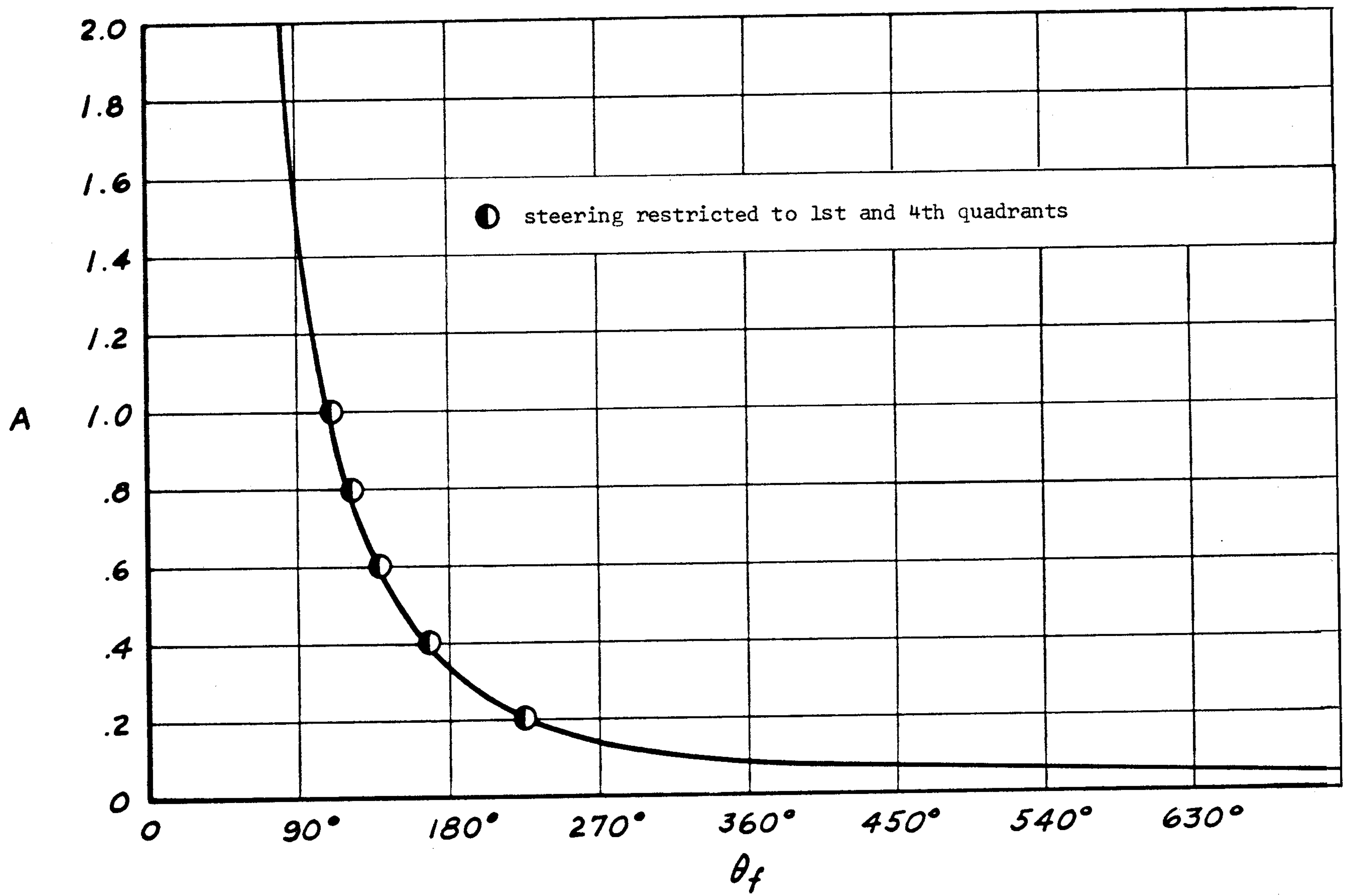


Figure 3.10 Thrust Level Coefficient vs. Terminal Range Angle for Minimum Angle Transfer

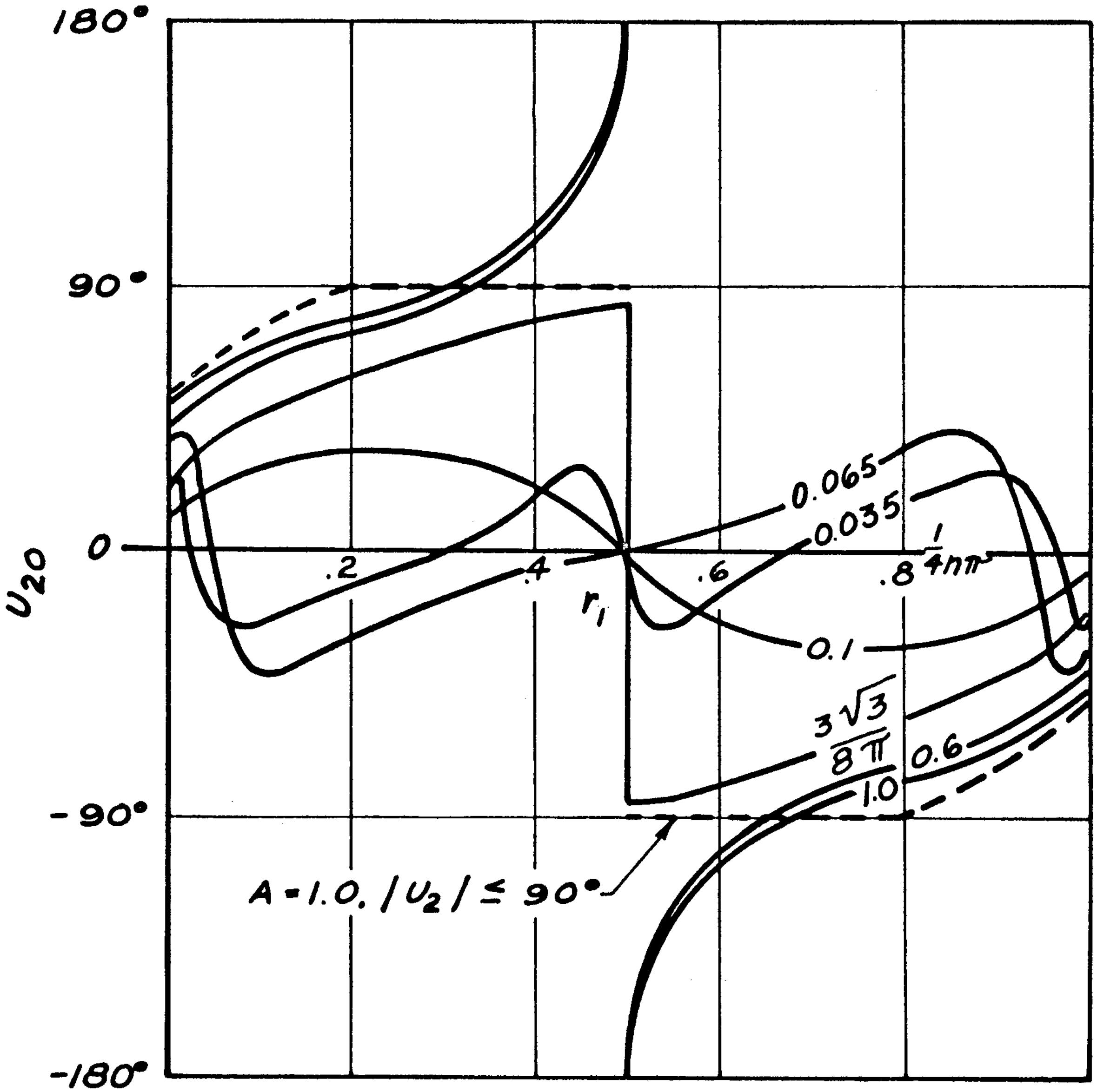


Figure 3.11 Steering Angle vs. Radius for Various Values of A in the Minimum Angle Transfer

$$A = .1 \,, \qquad A > \tfrac{1}{4}\pi$$
$$A = .065 \,, \qquad \tfrac{1}{8}\pi < A < \tfrac{1}{4}\pi$$
$$A = .035 \,, \qquad \tfrac{1}{2}\pi < A < \tfrac{1}{8}\pi$$

and that the number of zeros of u_z is one in the first case, two in the second case, three in the third, and so on.

Figure 3.12 is a plot of the fuel ratio, μ, against the final range angle. The fuel ratio is defined as the fuel required to achieve transfer divided by the fuel consumed in an impulsive Hohmann transfer with the same exhaust velocity (see Section 2.3.2.3 for a discussion of impulsive transfers and Hohmann ellipse). Reference 3.16 shows that the minimum value of μ for this case is one, therefore, referring to Figure 3.12 it is seen that for $\theta_f = 2N\pi, (A = 1/4\pi N)$, the fuel ratio is one; thus, the minimum angle transfer is also the minimum fuel transfer.

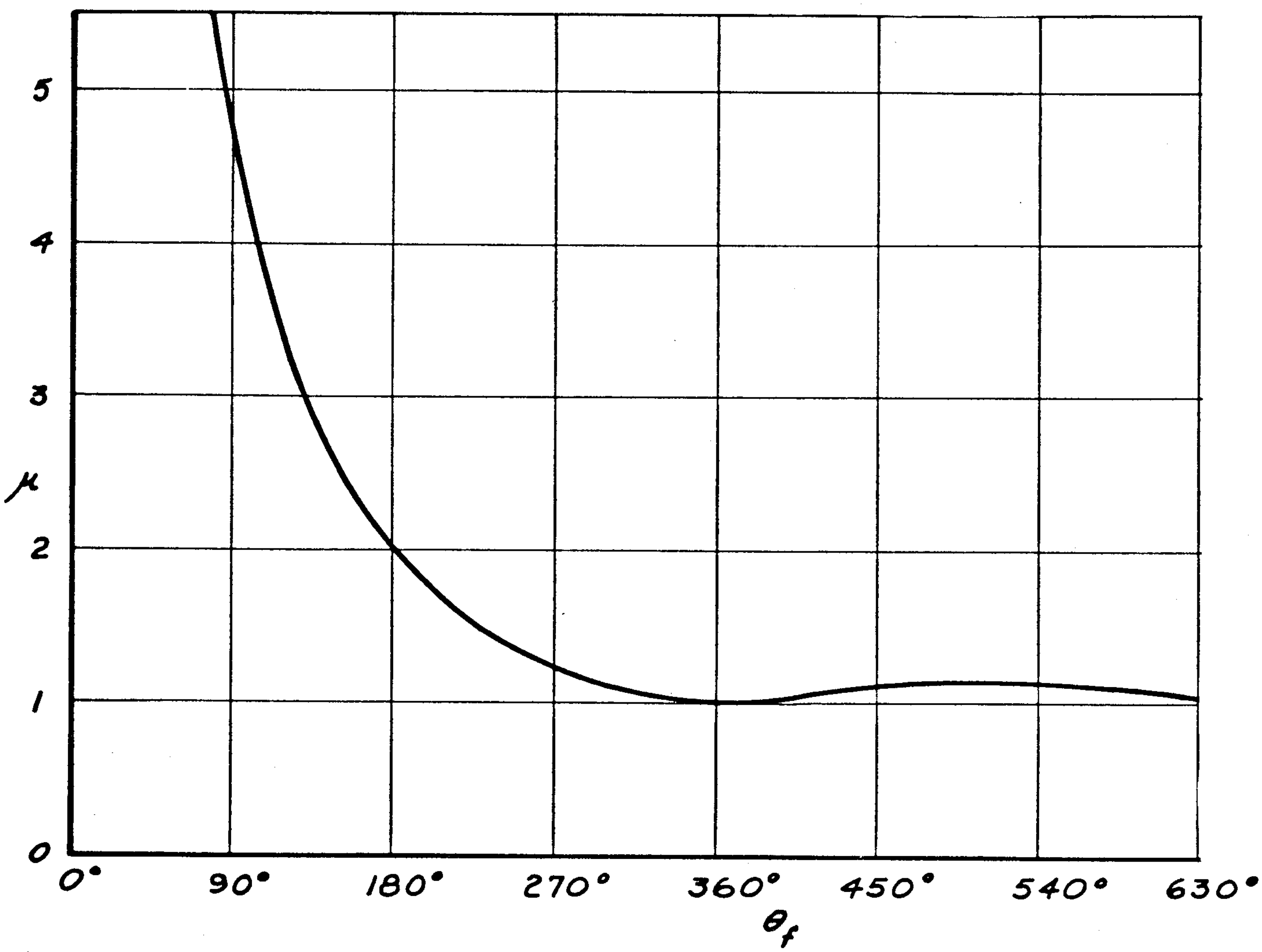


Figure 3.12 Fuel Ratio vs. Terminal Range Angle for Minimum Angle Transfer

As a second example, consider the minimum fuel transfer. For the case the optimization criteria is

$$\phi(x_f, \theta_f) = m_0 - m_f$$

and the boundary value on $P_4(\theta_f)$ is

$$P_4(\theta_f) = 1$$

The co-state variables for this case are given in terms of θ and the initial values of the variables as

$$
\begin{aligned}
P_1(\theta) &= P_1(0)\cos\theta + P_3(0)\sin\theta \\
P_2(\theta) &= -2P_1(\theta) + P_2(0) + 2P_1(0) \\
P_3(\theta) &= P_3(0)\cos\theta - P_1(0)\sin\theta \\
P_4(\theta) &= 1
\end{aligned}
$$

The minimum fuel transfer differs from the minimum angle transfer in that thrusting periods are interspersed with coasting periods. In Figure 3. 13 the fuel ratio is plotted against θ with A as a parameter.

From Figure 3. 13 it can be seen that the lower fuel limit ($\mu = 1$) can be achieved for any thrust level coefficient between 1. 0 and $1/4\pi$ if the value of θ_f is made large enough. The manner in which this limit is reached is the finite thrusting analog of the Hohmann transfer in that the maneuver consists of two equal burning periods in which the thrust is circumferential with centers of the burning periods separated by 180°. (See Figure 3. 15.)

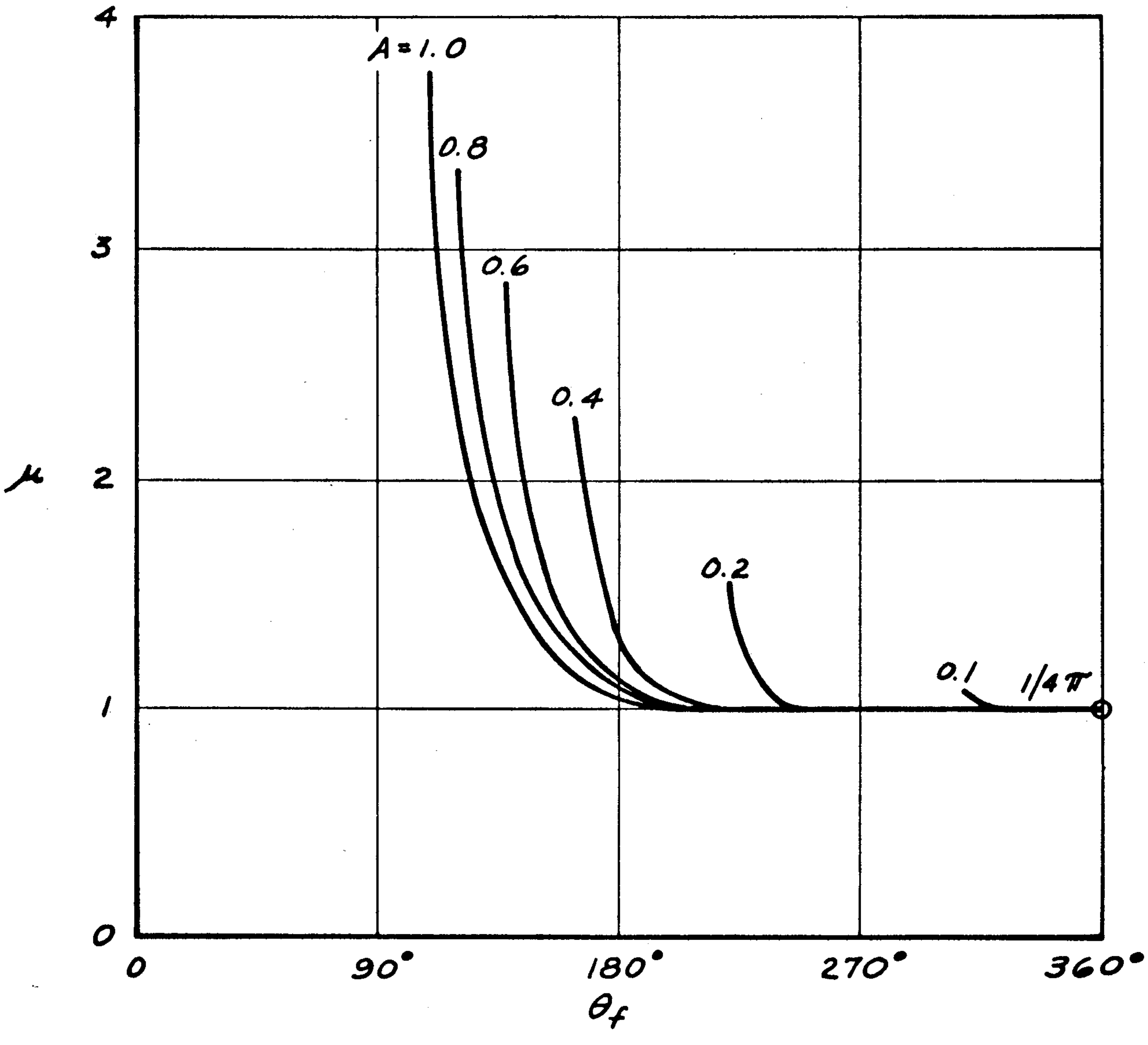


Figure 3.13 Fuel Ratio Vs. Terminal Range Angle For Minimum Fuel Transfer

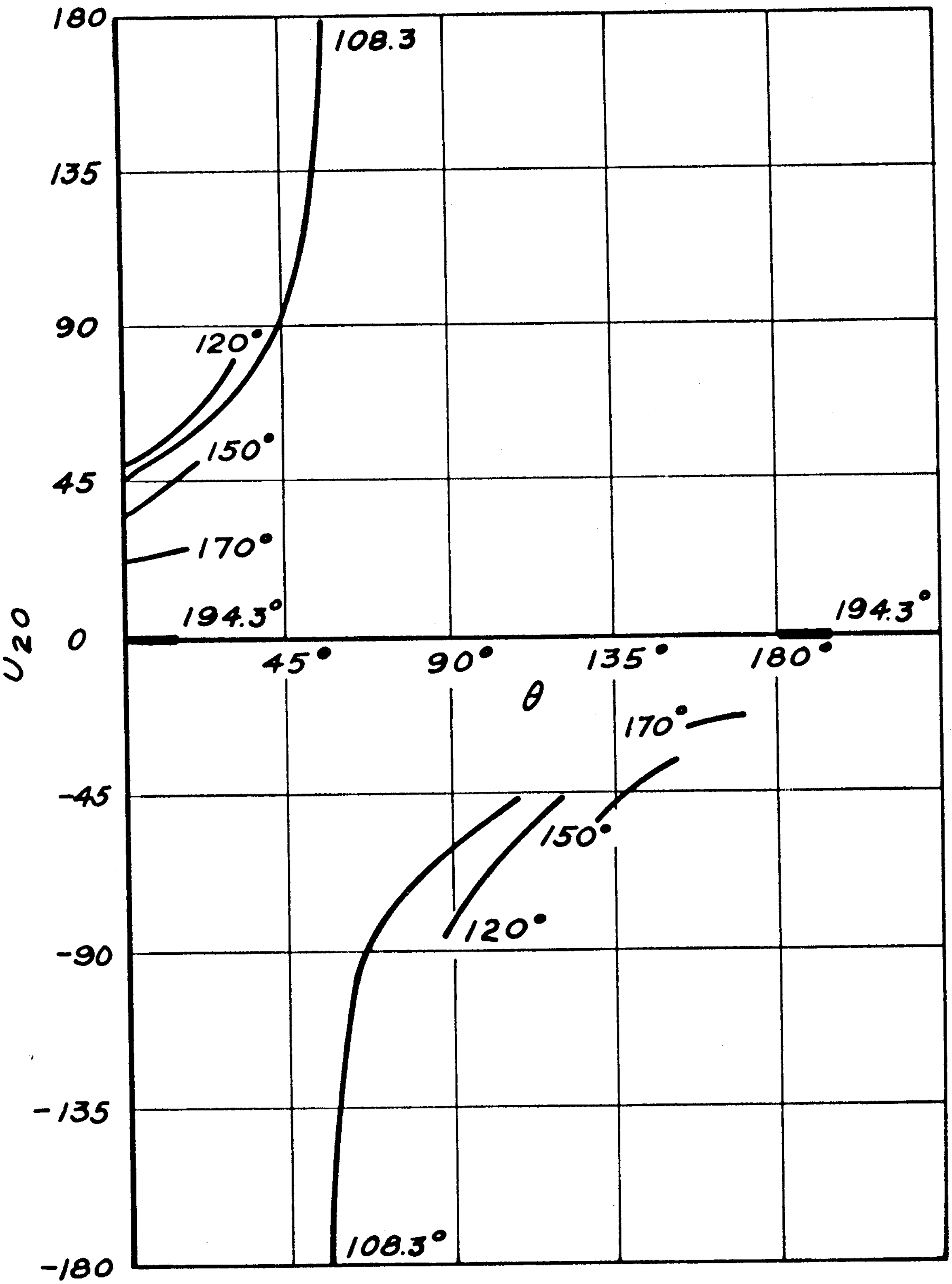


Figure 3.14 Steering Program for Minimum Fuel Transfer

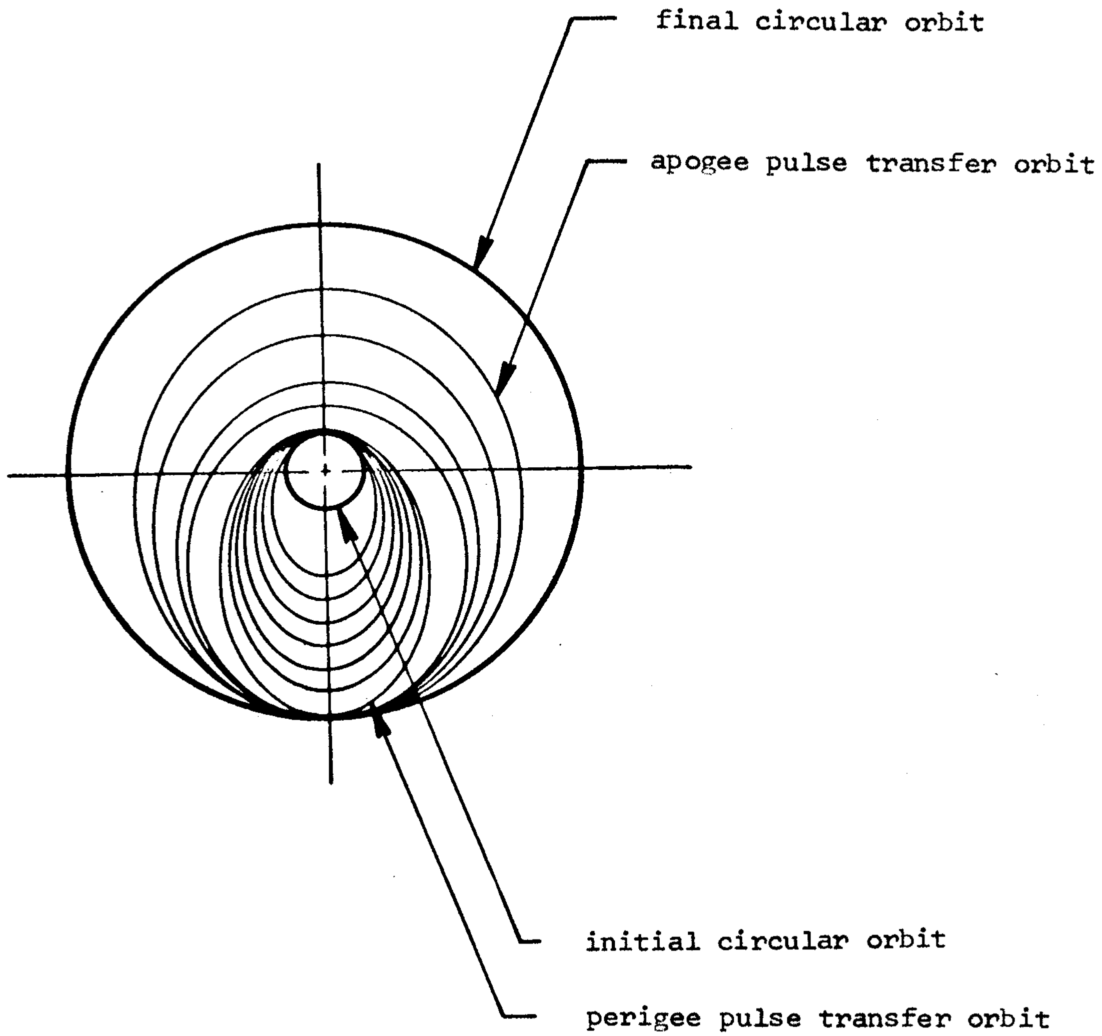


Figure 3.15 Finite Thrust Analog of Hohmann Transfer

2.3.2.2 Power Limited Vehicles

Some types of propulsion systems, such as electric or ion engines, are limited in the amount of power that can be supplied. For these systems, it is convenient to formulate the problem in terms of the power available as follows: The thrust of any rocket motor is given by (see Section 2.3.2.1)

$$T = \dot{m}c$$

(where $\dot{m}$ is the fuel flow rate and c is the rocket exhaust velocity) and the kinetic energy of the exhaust gas (or plasma, etc.) is

$$K.E. = \tfrac{1}{2} m c^2$$

Thus, since the rate of change of kinetic energy is the power P and since c is constant (i.e., the gas molecules are not accelerated relative to the rocket as they leave the rocket thrust chamber), p is given by

$$P = + \tfrac{1}{2} \dot{m} c^2 \qquad (3.93)$$

The thrust can now be expressed in terms of the power by combining equations (3.93) and (3.94) to give

$$T = \frac{2P}{c} \qquad (3.94)$$

and the fuel flow rate is obtained in terms of the power by rearranging equation (3.94)

$$\dot{m} = + \frac{2P}{c^2} \qquad (3.95)$$

The following section develops the equations for optimizing the power-limited system for a general cost function of the terminal state. The results will be applied to particular cost functions in Section 2.3.2.2.2. Finally, a linearized problem which yields an analytic solution will be treated in Section 2.3.2.2.3.

2.3.2.2.1 Optimization of a General Function of the Terminal State. The formulation of a problem which seeks to optimize some function of the terminal state using the Pontryagin Maximum Principle requires a knowledge of the exact form of the function to be optimized when obtaining boundary conditions. For this reason, the general function of the terminal state, denoted by $\phi(x, t_f)$, will be considered in this section; and characteristics of the solution, which are independent of the boundary condition, will be developed and discussed.

The material presented in this section is a review of Reference 3.13 with some modification in the notation so that continuity with other sections of this monograph is maintained. As in Section 2.3.2.1, the motion will be described in an inertial Cartesian (x, y, z) system whose origin is at the center of the force field. The components of the state vector are identical to those defined in Section 2.3.2.1, but are repeated for clarity

$$
\begin{array}{lll}
x_1 = x & x_4 = \dot{x} & x_7 = m \\
x_2 = y & x_5 = \dot{y} & \\
x_3 = x & x_6 = \dot{z} &
\end{array}
$$

The state equations will appear different because of the formulation of acceleration in terms of power. These equations are

$$
\begin{array}{lll}
\dot{x}_1 = x_4 & \dot{x}_4 = \dfrac{W}{c x_7} \ell_1 - g_1 & \dot{x}_7 = - \dfrac{W}{c^2} \\
\dot{x}_2 = x_5 & \dot{x}_5 = \dfrac{W}{c x_7} \ell_2 - g_2 & \\
\dot{x}_3 = x_6 & \dot{x}_6 = \dfrac{W}{c x_7} \ell_3 - g_3 &
\end{array}
\tag{3.96}
$$

where the ℓ_i's are direction cosines, where the g_i's are components of gravitational acceleration, and where W is the power which is limited by $0 \leq W \leq W_{MAX}$ (the notation for power has been changed at this point from P to W so that there will be no confusion between power and the components of the co-state vector). The control variables are the engine power, W, the exhaust velocity, c, and the direction cosines of the thrust direction ℓ_i. In terms of these variables, the hamiltonian for the problem is

$$
\begin{aligned}
H = {} & P_1 x_4 + P_2 x_5 + P_3 x_6 + P_4 \left(\frac{W}{c x_7} \ell_1 - g_1 \right) + P_5 \left(\frac{W}{c x_7} \ell_2 - g_2 \right) \\
& + P_6 \left(\frac{W}{c x_7} \ell_3 - g_3 \right) - \frac{P_7 W}{c^2}
\end{aligned}
\tag{3.97}
$$

Before the hamiltonian can be maximized with respect to the control variables, the constraints on these variables must be considered. The power, W, is constrained to the range $0 \le W \le W_{MAX}$; however, since the power appears linearly in H no problem is presented. The exhaust velocity, C, can also be bounded above and below. This constraint can be satisfied by defining a new variable γ by the equation

$$\gamma^2 - (c_{MAX} - c)(c - c_{MIN}) = 0$$

and adjoining the result to the hamiltonian by the use of an undetermined Lagrange multiplier λ. Similarly, the direction cosines, ℓ_i, are constrained by the relation

$$\ell_1^2 + \ell_2^2 + \ell_3^2 = 1$$

and by the use of another Lagrange multiplier, this relation can be adjoined to the hamiltonian. The augmented hamiltonian $\underset{\sim}{H}$ thus becomes (dropping terms which do not contain the control variables)

$$\underset{\sim}{H} = (P_4 \ell_1 + P_5 \ell_2 + P_6 \ell_3) \frac{W}{C X_7} - P_7 \frac{W}{C^2} + \lambda_1 \left[\gamma^2 - (C_{MAX} - C)(C - C_{MIN})\right] + \lambda_2 \left[\ell_1^2 + \ell_2^2 + \ell_3^2 - 1\right]$$

Since the hamiltonian is linear in the power W, it is maximized by choosing $W = 0$ or $W = W_{MAX}$, depending on the sign of the switching function S, is given by

$$s = \frac{1}{c x_7} (\ell_1 P_4 + \ell_2 P_5 + \ell_3 P_6) - \frac{P_7}{c^2}$$

That is

$$W \begin{cases} 0 & s < 0 \\ W_{MAX} & s > 0 \end{cases}$$

Now, maximization of H is accomplished by setting the derivatives with respect to the remaining control variables to zero, i.e.,

$$\frac{\partial H}{\partial c} = -\frac{W}{c x_7}(P_4 \ell_1 + P_5 \ell_2 + P_6 \ell_3) + 2\frac{P_7 W}{c^3} + \lambda_1 (2c - c_{MIN} - c_{MAX}) = 0$$

$$\frac{\partial H}{\partial \ell_1} = \frac{P_4 W}{c x_7} + 2\lambda_2 \ell_1 = 0$$

$$\frac{\partial H}{\partial \ell_2} = \frac{P_5 W}{c x_7} + 2\lambda_2 \ell_2 = 0$$

$$\frac{\partial H}{\partial \ell_3} = \frac{P_6 W}{c x_7} + 2\lambda_2 \ell_3 = 0$$

$$\frac{\partial H}{\partial r} = 2\gamma\lambda_1 = 0$$

(3.98)

If the first equation of (3.98) is solved for λ_1, the constraint equation for C solved for γ, and these two results combined in the last equation of (3.98) the result is

$$(C_{MAX} - C)(C - C_{MIN})\left[\frac{P_4 \ell_1 + P_5 \ell_2 + P_6 \ell_3}{x_7} - \frac{2P_7}{C}\right] = 0$$

From this equation, it is seen that C has three possible values given by

$$c = c_{MAX}$$
$$c = c_{MIN}$$
$$c = \frac{2P_7 x_7}{P_4 \ell_1 + P_5 \ell_2 + P_6 \ell_3}$$

Examination of the second derivative of H with respect to C may yield the answer to which of these three values is chosen. Reference 3.13 points out that for a typical Earth-Mars flight, the trajectory initiates at the lower limit, $C = C_{min}$. When conditions become appropriate, a change is made to the variable C mode during which the thrust velocity increases until it reaches its upper limit $C = C_{max}$. During the interval of $C = C_{max}$, there may or may not be a coast period called for. After some time, the conditions will become such that a return to the variable thrust mode is desired. The trajectory may end in this mode or return to the constant thrust mode with $C = C_{min}$.

Now λ_z can be eliminated from the second, third, and fourth equation of (3.98) by substitution of the constraint relation for the ℓ_i's with the result that the direction cosines can be expressed in terms of the co-state variables as

$$\ell_i = \frac{P_{i+3}}{\sqrt{P_4^2 + P_5^2 + P_6^2}} \qquad i = 1,3$$

These relations can be substituted into the expressions for the switching function and the variable exhaust. The resulting control is summarized below:

$$\ell_1 = \frac{P_4}{\sqrt{P_4^2 + P_5^2 + P_6^2}} \qquad \ell_2 = \frac{P_5}{\sqrt{P_4^2 + P_5^2 + P_6^2}} \qquad \ell_3 = \frac{P_6}{\sqrt{P_4^2 + P_5^2 + P_6^2}}$$

$$c = c_{MAX},\ c_{MIN},\ \text{or}\ 2 P_7 x_7 (P_4^2 + P_5^2 + P_6^2)^{-1/2}$$

$$W = \begin{cases} 0 & s < 0 \\ W_{MAX} & s > 0 \end{cases} \qquad \text{where} \quad s = \frac{\sqrt{P_4^2 + P_5^2 + P_6^2}}{c x_7} - \frac{P_7}{c^2}$$

To complete the discussion of this section, the co-state equations given by

$$\dot{P}_i = -\frac{\partial H}{\partial x_i}$$

where H is the original hamiltonian given by equation (3.97) leads to

$$\dot{P}_1 = \frac{\partial g_1}{\partial x_i} \qquad \dot{P}_4 = P_1 \qquad \dot{P}_7 = \frac{W}{c x_7^2}(P_4 \ell_1 + P_5 \ell_2 + P_6 \ell_3)$$

$$\dot{P}_2 = \frac{\partial g_2}{\partial x_2} \qquad \dot{P}_5 = -P_2$$

$$\dot{P}_3 = \frac{\partial g_3}{\partial x_3} \qquad \dot{P}_6 = -P_3$$

This set of differential equations plus the set given by equation (3.96) constitutes a total of 14 first-order differential equations which must be solved simultaneously. At the initial time, the seven elements of the state vector corresponding to position and velocity in the terminal orbit are known. Thus, a total of 13 boundary conditions are known and one more is required before a solution can be obtained. The final boundary condition cannot be specified until the optimization criterion has been specified. This subject is discussed in the next section.

A slight modification of this procedure is necessary if a constant exhaust velocity (i.e., $c_{MAX} = c_{MIN} = c$) is required, but the constant value is to be chosen for optimum performance. In this case, c can be included as a component of the state vector with the differential constraint

$$\dot{c} = 0$$

and unspecified boundary conditions. The first equation of (3.98) is now meaningless since c is constant, but an additional co-state equation similar to the first equation of (3.98) is obtained as

$$\dot{P}_8 = \frac{W}{c^2 x_7} (P_4 \ell_1 + P_5 \ell_2 + P_6 \ell_3) - 2 \frac{P_7 W}{c^3}$$

Since there are no boundary conditions specified for c , the boundary conditions on P_8 are

$$P_8(t_o) = P_8(t_f) = 0$$

2.3.2.2.2 Optimization Criteria. In this section three optimization criteria are considered and the corresponding boundary conditions developed which complete the statement of the problem given in the last section. The three criteria considered are: (1) final mass; (2) the integral of acceleration squared; and (3) minimum flight time for a given final mass. Reference 3.13 indicates that all three of these criteria are simultaneously maximized if there are no coast periods. If, however, the exhaust velocity is required to be constant (i.e., $c_{MAX} = c_{MIN} = c$), then the solution will contain coast intervals of different lengths depending upon which criterion function is selected.

The general expression for the boundary conditions on the co-state variables at the terminal time is

$$P_i(t_f) + \sum_j \alpha_j \frac{\partial \psi_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0 \qquad (3.99)$$

where the ψ_i's are the terminal constraints, $\phi(x_i, t_f)$ is the function of the terminal state to be optimized, and the α_i's are undetermined constraints. For the orbit transfer problem, the terminal constraints are of the form

$$\psi_i(x_1, x_2, x_3, x_4, x_5, x_6) = 0$$

Therefore,

$$\frac{\partial \psi_i}{\partial x_7} = 0$$

Consider the case where the terminal mass is to be maximized. In this problem, $\phi(x_i, t_f) = x_7$ and $P_7 = -1$ supply the additional boundary conditions.

Next, consider minimization of the integral of acceleration squared. From equations (3.93) and (3.94), the following relation is obtained

$$\int_{t_0}^{t} a^2\, dt = \int_{t_0}^{t} \frac{4W^2}{c^2 m^2} = \int_{t_0}^{t} -\frac{2W\dot{m}}{m^2}\, dt$$

Assuming no singular arcs, W assumes only its maximum value or zero and can, therefore, be taken through the integral giving (for the case of no coast periods)

$$\int_{t_0}^{t} a^2\, dt = 2W_{MAX}\left[\frac{1}{m(t_f)} - \frac{1}{m(t_0)}\right]$$

Thus, minimization of the integral of acceleration squared is equivalent to maximizing the final mass. The boundary condition for $P_7(t_f)$ is obtained from equation (3.99) as

$$P_7(t_f) = \frac{-2W_{MAX}}{m^2(t_f)}$$

Although $m(t_f)$ is not known, it will not be zero; therefore, $P_7(t_f)$ is nonzero. This fact allows multiplication of the co-state variables by an arbitrary scaling factor, so that this condition can be reduced to

$$P_7(t_f) = -1$$

without loss of generality.

For minimum time problems with the final mass specified, the value of the final mass provides the fourteenth boundary condition.

For these three optimization criteria, a total of 14 boundary conditions have been found for the 14 first-order differential equations comprising the state and co-state systems. In all cases, the problem is of the two-point boundary-value type and numerical techniques required to obtain a solution.

2.3.2.2.3 Linear Power-Limited Systems. If the equations of motion are linearized and the cost function is the integral of acceleration squared, then an analytic solution to the power-limited optimum transfer problem exists. However, it is not only the linearization of equations of motion, but also the form of the cost function which allows the analytic solution. In the last section, it was shown that minimization of the integral of acceleration squared was equivalent to maximizing the final mass - a feature which distinguishes the power-limited problems from thrust-limited problems where the equivalence is not valid. It is not implied, however, that an analytic solution to the thrust-limited problem with a similar cost function is impossible. Rather, that minimizing the integral of the acceleration squared has more physical significance (in most cases) for power-limited system than for thrust-limited systems. The following development of the solution is a review of References 3.14 and 3.15.

In order to obtain an analytic solution, the problem will be formulated in a slightly different way. The principle difference will consist of defining a new state variable as

$$x_7(t) = \int_{t_0}^{t} \frac{a^2}{2}\, dt \quad , \quad \dot{x}_7 = \frac{a^2}{2}$$

where a is the acceleration magnitude. This variable will be used in lieu of the mass-flow equation of the last section. The linearized gravity was developed in Section 2.3.1. Thus, the only difference between the formulation of Section 2.3.1 and that presented here is that the coordinate system is assumed to be located on the initial orbit, instead of the final one. The control vector has components defined by

$$u_1 = \sqrt{\frac{W_0}{2}}\, A_x$$

$$u_2 = \sqrt{\frac{W_0}{2}}\, A_y \qquad a^2 = W_0 \left(A_x^2 + A_y^2 + A_z^2\right)$$

$$u_3 = \sqrt{\frac{W_0}{2}}\, A_z$$

where W_0 is a constant, and A_i contains the variable exhaust velocity and mass. It is assumed that there are no constraints on the exhaust velocity. Therefore, the linearized state equations are (in terms of the nondimensional time parameter $\tau = \omega t$)

$$\begin{aligned}
\dot{x}_1 &= x_4 & \dot{x}_4 &= A_x + 2x_5 & \dot{x}_7 &= \frac{W_0}{2}\left(A_x^2 + A_y^2 + A_z^2\right) \\
\dot{x}_2 &= x_5 & \dot{x}_5 &= A_y + 3x_2 - 2x_4 & & \\
\dot{x}_3 &= x_6 & \dot{x}_6 &= A_z - x_3 & &
\end{aligned}$$

and hamiltonian for this problem is

$$\begin{aligned}
H = P_1 x_4 + P_2 x_5 + P_3 x_6 &+ P_4 (A_x + 2x_5) + P_5 (A_y + 3x_2 - 2x_4) \\
&+ P_6 (A_z - x_3) + P_7 \frac{W_0}{2}\left(A_x^2 + A_y^2 + A_z^2\right)
\end{aligned}$$

Thus, it follows that the co-state equations are

$$\begin{aligned}
\dot{P}_1 &= 0 & \dot{P}_4 &= -P_1 + 2P_5 & \dot{P}_7 &= 0 \\
\dot{P}_2 &= -3P_5 & \dot{P}_5 &= -P_2 - 2P_4 & & \\
\dot{P}_3 &= P_6 & \dot{P}_6 &= -P_3 & &
\end{aligned}$$

Now, maximizing the hamiltonian with respect to A_X, A_Y and A_Z by setting the derivatives with respect to these parameters equal to zero yields the three equations

$$P_4 = -P_7 \frac{W_o}{2} A_x$$

$$P_5 = -P_7 \frac{W_o}{2} A_y$$

$$P_6 = -P_7 \frac{W_o}{2} A_z$$

However, the boundary condition for P_7 is

$$P_7(t_f) = -\frac{\partial \phi}{\partial x_7} = -1$$

Therefore,

$$A_x = \frac{2P_4}{W_o} \qquad A_z = \frac{2P_6}{W_o}$$

$$A_y = \frac{2P_5}{W_o}$$

Now, the co-state equations can be integrated to obtain a representation of the variables in terms of the time parameter τ and six constants of integration as

$$P_1 = W_o c_o$$

$$P_2 = -6W_o(c_4 + c_o\tau - c_1 \cos\tau + c_2 \sin\tau)$$

$$P_3 = 2W_o(c_5 \sin\tau + c_3 \cos\tau)$$

$$P_4 = W_o(3c_4 + 3c_o\tau - 4c_1 \cos\tau + 4c_2 \sin\tau)$$

$$P_5 = 2W_o(c_o + c_1 \sin\tau + c_2 \cos\tau)$$

$$P_6 = 2W_o(c_5 \cos\tau - c_3 \sin\tau)$$

These equations, in turn, can be substituted into the expression for the control and the resulting control expressions can be substituted into the state equations and integrated to yield the following relations

$$X = \left[16(1-\cos\tau) - 10\tau\sin\tau\right]c_1 + \left[22\sin\tau - 10\tau\cos\tau - 12\tau\right]c_2 - \left[\frac{9}{2}\tau^2 - 12(1-\cos\tau)\right]c_4$$

$$y = \left[5(\sin\tau - \tau\cos\tau)\right]c_1 + \left[5\tau\sin\tau - 8(1-\cos\tau)\right]c_2 + \left[6(\sin\tau - \tau)\right]c_4$$

$$z = \left[\tau\cos\tau - \sin\tau - (\tau\sin\tau/\sin\tau_f)\right]c_3$$

$$\dot{X} = \left[6\sin\tau - 10\tau\cos\tau\right]c_1 + \left[10\tau\sin\tau - 12(1-\cos\tau)\right]c_2$$

$$\dot{y} = \left[5\tau\sin\tau\right]c_1 + \left[5\tau\cos\tau - 3\sin\tau\right]c_2 - \left[3(1-\cos\tau)\right]c_4$$

$$\dot{z} = -\left\{\tau\sin\tau + \left[(\sin\tau + \tau\cos\tau)/\tan\tau_f\right]\right\}c_3$$

Thus, the constants can be evaluated from the boundary conditions on the state vector.

2.3.2.3 Impulsive Transfer

In many actual orbit transfer problems, the length of time that the rocket motors are required to operate is very small compared to the coasting times. For these problems, therefore, to a good approximation, the thrust can be regarded as impulsive; and the motion during periods of maximum thrust can be neglected. Thus, the optimum trajectory is composed of null thrust arcs meeting at junction points where velocity impulses are applied. In this case, the optimization problem reduces to one of optimizing a function of several variables rather than functionals as in the finite case.

For the case of co-planar orbits, the theory of Section 2.3.2.2 can be used, with modification, to yield a set of transcendental equations which must be satisfied by an optimum transfer. This problem is discussed in Section 2.3.2.3.1 for the general case of elliptic initial and final orbits and for the case of nearly-circular orbits. If the initial and final orbits are not co-planar, then a set of equations such as developed for the co-planar case is not available. The solution of this three-dimensional optimization problem is obtained by application of a numerical technique for locating the extremals of functions of several variables. This problem is discussed in Section 2.3.2.3.2.

2.3.2.3.1 Co-Planar Transfers. In order to develop a set of equations whose solution compromises the optimum impulsive transfer between two co-planar orbits, a further examination of the theory developed in Section 2.3.2.1 must be made at points where the impulses are applied. Following Lawden (Reference 3.8), a period of thrust is considered which is short enough to allow the components of the primer vector to be considered constant over that interval. In this case, equation (3.77) can be integrated over the interval as

$$P_7(t+\Delta t) = P_7(t) + \sqrt{P_4^2+P_5^2+P_6^2}\int_t^{t+\Delta t} \frac{c\beta}{X_7^2}\,dt$$

and since $\beta = \frac{dx}{dt}$ this expression becomes

$$P_7(t+\Delta t) = P_7(t) + \sqrt{P_4^2+P_5^2+P_6^2}\left[\frac{c}{X_7(t+\Delta t)} - \frac{c}{X_7(t)}\right]$$

But from equation (3.78) (which must hold at points of discontinuity of thrust), it is seen that this equation will be valid only if

$$\sqrt{P_4^2+P_5^2+P_6^2} = 1 \tag{3.100}$$

This equation implies that the components of the primer vector are themselves direction cosines for the thrust.

Before going further with the development of the equations for optimum impulsive transfer, it is necessary to discuss the number of impulses which will be used to affect the transfer. If the orbits intersect, then transfer can take place as a result of a single impulse applied at the point of intersection. However, if the orbits do not intersect, then at least two impulses must be used; furthermore, it is found that in some cases where the orbits do intersect, a two-impulse transfer produces a more optimum transfer than a single impulse. (This point is discussed in more detail later.) Unfortunately, the theory does not provide a means for determining the number of impulses which will yield the true optimum solution; this number can be determined only by direct comparison of the results of the optimization problem for different numbers of impulses. Therefore, in the development to follow only two impulse transfers are considered.

Now, consider two co-planar elliptic orbits described by: (1) the semi-major axis, a ; (3) the eccentricity, e ; and (3) the argument of perigee, w , and let the subscripts 1, 2 (on the variables a , e , w) denote the initial and final orbits, respectively. The problem is to define the optimum transfer for the case where the time of transit (the constant C in equation (3.86) is zero) is unspecified. Further, since the transfer orbit is confined to the plane of the initial and final orbits, the constants E and F of equation (3.86) are taken as zero. Now, let the angle ϕ be the angle between the thrust direction, and the γ axis and ν be the direction of the radius vector from the perigee direction.

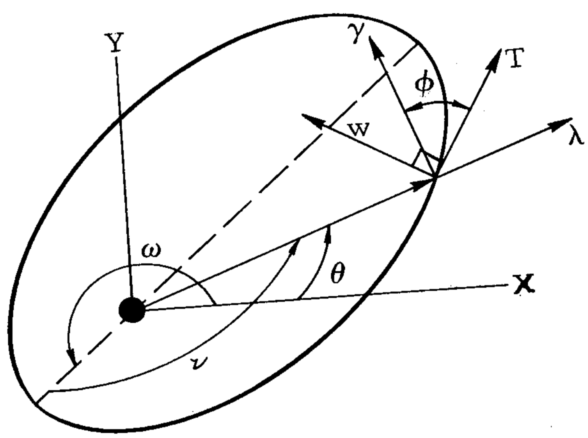


Figure 3.16 Coordinate Definition for Impulsive Co-planar Transfer

Now, the primer vector (equation (3.86)) and it's derivative must be continuous at the point at which the spacecraft is injected into the transfer orbit. This fact and equation (3.86), and the condition that the components of the primer vector are the direction cosines for the thrust vector, yield

$$-A'\cos\nu' + B'e'\sin\nu' = A_1\cos\nu_1 + B_1 e_1\sin\nu_1 = \sin\phi_1$$

$$-A'\sin\nu' + B'(1+e'\cos\nu') + \frac{D'-A'\sin\nu'}{1+e'\cos f'} = -A_1\sin\nu_1 + B_1(1+e_1\cos\nu_1) + \frac{D_1-A_1\sin\nu}{1+e_1\cos\nu_1}$$

$$\sqrt{a'}\left(\frac{A'\sin\nu'-D'}{1+e'\cos\nu'} - B\right) = \sqrt{a_1}\left(\frac{A_1\sin\nu_1-D_1}{1+e_1\cos\nu_1} - B_1\right)$$

$$(a')^{-3/2}\left[-A'(e'+\cos\nu') + D'e'\sin f'\right] = a_1^{-3/2}\left[-A_1(e_1+\cos\nu_1 + D_1 e_1\sin\nu_1\right]$$

(3.101)

where the primes denote constants on the transfer orbit. Thus, if the velocity components along the directions λ and γ are V_λ, V_γ, equation (3.83) can be applied at the injection point to give

$$e'\cos\nu' = \frac{a'}{r} - 1$$

$$e_1\cos\nu_1 = \frac{a_1}{r} - 1$$

$$V_\gamma = \sqrt{\frac{\mu}{a}}\,(1+e_1\cos\nu_1)$$

(3.102)

Similarly, if the velocity component normal to the thrust is W, then

$$W = V_\gamma\sin\phi - V_\lambda\cos\phi$$

and again using (3.83), this equation becomes

$$e' \sin \nu' = \frac{a'}{r} \tan \phi - W\sqrt{\frac{a'}{\mu}} \sec \phi \tag{3.103}$$

But, since W is unaffected by the thrust, this equation is also valid in the initial orbit

$$e_1 \sin \nu_1 = \frac{a_1}{r} \tan \phi - W\sqrt{\frac{a_1}{\mu}} \sec \phi \tag{3.104}$$

Thus, A_1, B_1, D_1', B_1', D_1' can be eliminated from equation (3.101) to yield A_1' (using equations (3.102) through (3.104) as

$$A_1 = \sqrt{\frac{a^{-1}}{\mu (e')^2}}\left(W - \frac{\mu}{W r} \sin^2 \phi \right)$$

Therefore, using (3.102) through (3.104) and this expression for A_1, the second equation of (3.101) gives

$$B' = \frac{1}{(e')^2}\left\{ \left(\frac{1}{a'} - 1 \right)\left(1 + \sqrt{\frac{\mu}{W^2 a'}} \sin \phi \right) \cos \phi + \left(\frac{a'}{r} \sin \phi - W\sqrt{\frac{a_1'}{\mu}} \right) \tan \phi \right\}$$

Finally, these expressions for A' and B' in the fourth equation of (3.101) gives

$$B' + D' = -\left[1 + \sqrt{\frac{\mu}{W^2 a'}} \left(1 + \frac{a'}{r} \right) \sin \phi \right] \cos \phi$$

Manipulations similar to these can be performed at the point where the second impulse is applied, and since the constants for the transfer orbit are common to both sets of equations, they can be eliminated. If the following variable changes are made

$$s \triangleq \frac{1}{r}$$

$$z \triangleq W\,(\sqrt{\mu}\,\sin\phi)^{-1}$$

$$\ell \triangleq \frac{1}{a}$$

$$q \triangleq \frac{e}{a}$$

the resulting set (independent of A', B', D', A_1, B_1, D_1, A_2, B_2, and D_2) is (see Reference 3.8 for details)

$$\left(z_1 - \frac{s_1}{z_1}\right)\sin\phi = \left(z_2 - \frac{s_2}{z_2}\right)\sin\phi_2$$

$$(s_1 - \ell')\left[1 + \frac{(\ell')}{z_1}\right]\cos\phi_1 + \left[s_1 - z_1(\ell')^{1/2}\right]\sin\phi_2\,\tan\phi_2$$
$$= (s_2 - \ell')\left[1 - \frac{(\ell')^{1/2}}{z_2}\right]\cos\phi_2 + \left[s_2 - z(\ell')^{1/2}\right]\sin\phi_2\,\tan\phi_2$$

$$\left[1 + \frac{s_1 + \ell'}{z_1(\ell')^{1/2}}\right]\cos\phi_1 = \left[1 + \frac{s_2 + \ell'}{z_2(\ell')^{1/2}}\right]\cos\phi_2$$

(3.105)

The results from the equations (3.102) through (3.104) evaluated at each impulse point are

$$
\begin{aligned}
q_1 \cos \nu_1 &= s_1 - \ell_1 \\
q_1 \sin \nu_1 &= \left(s_1 - z_1 \sqrt{\ell_1}\right) \tan \phi_1 \\
q' \cos \nu' &= s_1 - \ell' \\
q' \sin \nu' &= \left(s_1 - z_1 \sqrt{\ell'}\right) \tan \phi_1 \\
q_2 \cos \nu_2 &= s_2 - \ell_2 \\
q_2 \sin \nu_2 &= \left(s_2 - z_2 \sqrt{\ell_2}\right) \tan \phi_2 \\
q' \cos \nu_2' &= s_2 - \ell' \\
q' \sin \nu_2' &= \left(s_2 - z_2 \sqrt{P'}\right) \tan \phi_2 \\
\nu_2' &= \nu_2 + \omega_2 - \nu_1 - \omega_1 + \nu_1'
\end{aligned}
\tag{3.106}
$$

Thus, the set of twelve equations given by (3.105) and (3.106) can be solved for the twelve unknowns.

$$
S_1,\ S_2,\ \nu_1,\ \nu_2,\ \nu^1,\ \nu_2^{\,1},\ Z_1,\ Z_2,\ \phi_1,\ \phi_2,\ \ell' \ \text{and}\ q'
$$

and the optimization has been accomplished.

This set of twelve equations seems formidable and is, in fact, difficult to solve. However, a great deal of simplification results if the orbits between which the transfer is being made are nearly circular. In this case, the parameters, ℓ , s , and z can be expanded in a power series. Such an expression is carried out by Lawden in Reference 3.8. A portion of Lawden's discussion, the case when only the zero-order terms are retained, is presented below.

If the parameters ℓ , s , k_2 , and s_2 are assumed constant (corresponding to circular orbits), the set of equations (3.106) becomes

$$
\begin{aligned}
0 &= s_1 - \ell_1 \\
0 &= \left(s_1 - z_1\sqrt{\ell_1}\right)\tan\phi_1 \\
g'\cos\nu' &= s_1 - \ell \\
g'\sin\nu' &= \left(s_1 - z_1\sqrt{\ell'}\right)\tan\phi_1 \\
0 &= s_2 - \ell_2 \\
0 &= \left(s_2 - z_2\sqrt{\ell_2}\right)\tan\phi_2 \\
g'\cos\nu_2' &= s_2 - \sqrt{\ell'} \\
g\cos\nu_2' &= \left(s_2 - z_2\sqrt{\ell'}\right)\tan\phi_2 \\
s_1 &= z_1\sqrt{\ell_1}\,, \quad s_2 = z_2\sqrt{\ell_2}
\end{aligned}
\tag{3.107}
$$

But, if the second and fifth equation are satisfied by taking

$$s_1 = z_1 \sqrt{\ell_1}, \quad s_2 = z_2 \sqrt{\ell_2}$$

unacceptable conditions on ℓ_1 and ℓ_2 result. Therefore, the only possibility is that

$$\tan\phi_1 = \tan\phi_2 = 0$$

Then, from the last four equations of (3.107), the angles ν_1 and ν_2 must either be zero or π. However, if both these angles are zero or π, the equations are contradictory. Therefore, take

$$\nu^1 = 0, \; \nu_2{}^1 = \pi \tag{3.108}$$

The last four equations of (3.107) now yield

$$
\begin{aligned}
\ell' &= \tfrac{1}{2}(s_1 + s_2) = \tfrac{1}{2}(\ell_1 + \ell_2) \\
g' &= \tfrac{1}{2}(s_1 - s_2) = \tfrac{1}{2}(\ell_1 - \ell_2) \qquad \ell_1 > \ell_2
\end{aligned}
$$

Equation (3.108) indicates that the impulsive thrust is applied at the two apses of the transfer ellipse, which is tangential to the circular-terminal orbits at these points. This mode of transfer between two circular orbits via cotangential ellipse was discovered by Hohmann and the ellipse is named after him.

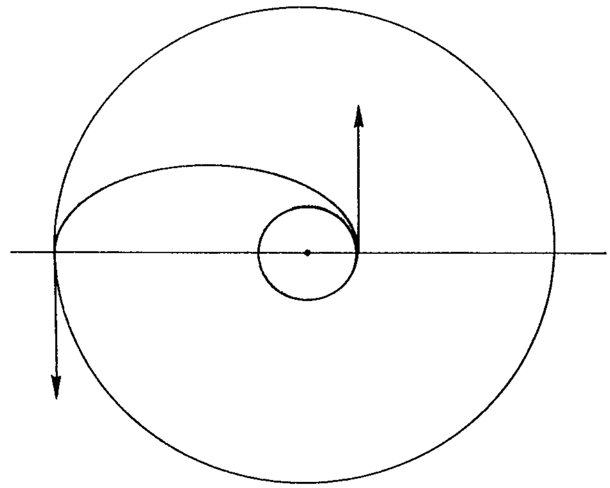


Figure 3.17 Hohmann Transfer

It turns out that the Hohmann transfer is not only a local minimum but a global minimum on the class of all possible trajectories between circular orbits not too greatly separated. However, Hoelker and Silber (Reference 3.17) demonstrated that for circular orbits of large separation, the Hohmann two-impulse transfer could be improved by a three-impulse maneuver termed a "bi-elliptical transfer." Such a transfer is shown in Figure 3.18.

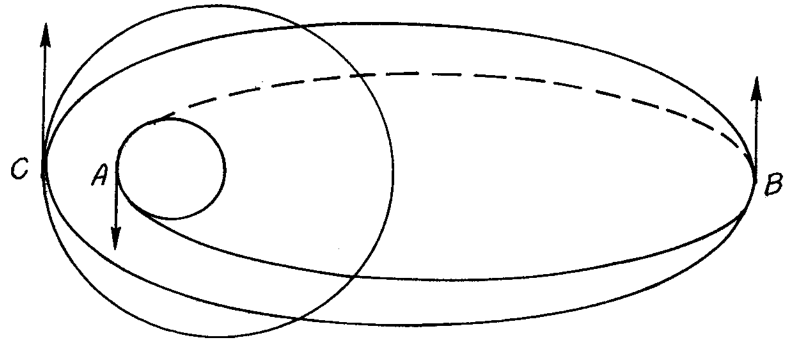


Figure 3.18 Bi-elliptic Transfer

The bi-elliptic transfer is accomplished by first transferring to the perigee of an elliptical orbit (point A, Figure 3.18), which has an apogee radius greater than the radius of the final orbit; next, a transfer to the apogee of a second elliptical orbit (point B, Figure 3.18) at the apogee of the first elliptical orbit; finally, a transfer is made to the terminal-circular orbit (point C, Figure 3.18) at the perigee of the second ellipse. The authors of Reference 3.17 further established that for $r_f/r_o \leq 11.9$ (r_f/r_o = ratio of the radius of the final circular orbit to that of the initial), the three-**burn** maneuver is never better than the Hohmann transfer. For r_f/r_o between 11.9 and 15.6 the three-bum maneuver represents an improvement, provided the point B in Figure 3.18 is sufficiently far removed from the final radius (with the best results occurring when B is at infinity). For $r_f/r_o > 15.6$, the three-**burn** maneuver is better than the Hohmann transfer for any location of the point B beyond the final orbital radius.

In 1960 L. Ting (Reference 3.18) proved that in transferring between any two co-planar elliptical orbits whose orientation is free orientation (i.e., there is no constraint on the location of perigee direction of either orbit), the two-impulse maneuver of the Hohmann type as shown in Figure 3.19 is locally minimizing.

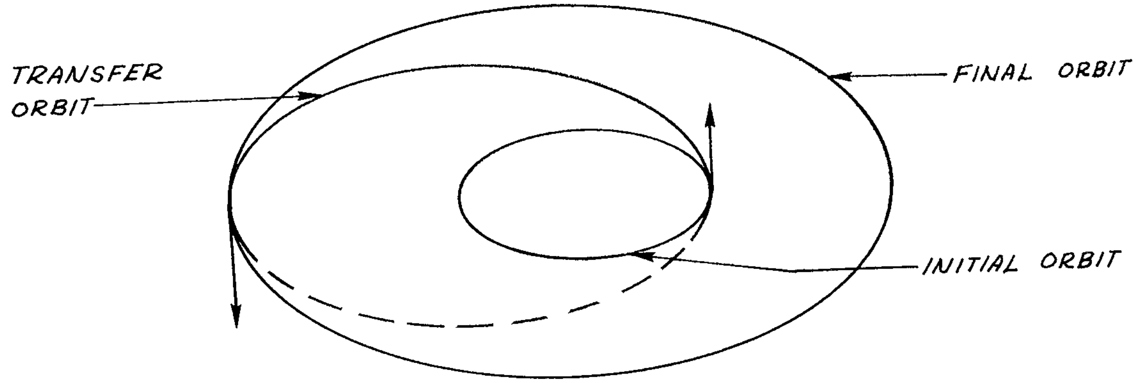


Figure 3.19 Hohmann Transfer Between Elliptical Orbits

Under the assumption that the Hohmann-type transfer is the absolute minimum two-impulse transfer, Ting was able to conclude in Reference 3.19 that under no circumstances could any improvement be had in the fuel consumption in transferring between elliptical orbits with free orientation by using more than three impulses; and that a three-impulse transfer is better than a Hohmann-type transfer only under certain circumstances which reduce to the condition developed by Hoelker and Silber when the orbits are circular. Ting's assumption that the Hohmann transfer is the global minimum two-impulse transfer was verified by R. Barrar (Reference 3.20). The combined results of References 3.17 to 3.20 established that the Hohmann impulsive transfer is the minimum fuel trajectory on the class of all impulsive transfers provided the ratio of the orbits r_f/r_o is less than 11.9.

McIntyre, in Reference 3.16, demonstrates that any finite burn transfer can be approximated as closely as desired by an impulsive vehicle with the same exhaust velocity. This work extends the class of trajectories over which the Hohmann maneuver is a global minimum to include continuous burn trajectories.

2.3.2.3.2 <u>Three-Dimensional Transfers</u>. The three-dimensional case is conveniently handled by discarding the theory developed in Section 2.3.2.3.1 in favor of performing an ordinary (numerical) optimization of a function of several variables. Such an approach is taken in References 3.21, 3.22, 3.25 and 3.27; the following discussion is a review of this work.

2.3.2.3.2.1 Determination of the Functions to be Minimized. The function to be optimized (the "characteristic velocity," denoted by ΔV_c) is defined as the absolute value of the difference between the velocity of the initial and final orbits at the point of intersection; one or both of these may be an intermediate transfer orbit. In the case where more than one impulse is assumed, the characteristic velocity is defined as the sum of the characteristic velocities corresponding to each impulse, and the number of impulses to achieve the optimum transfer must be obtained by examination of the characteristic velocity for several corrective schemes. Since two-body orbits are more easily described in terms of the classible orbital elements, the expression for the characteristic velocity for a two-impulse transfer will be developed.

The two orbits between which a transfer is to occur can be defined by their semi-latus rectum (p), their eccentricities (e), the inclination of one orbit plane with respect to the other (i), the direction of perigee (w) measured from the line of intersection of the two orbit planes, and the position on the orbit at which the transfer is to take place measured from the same reference (φ). This geometry is shown in the following sketch.

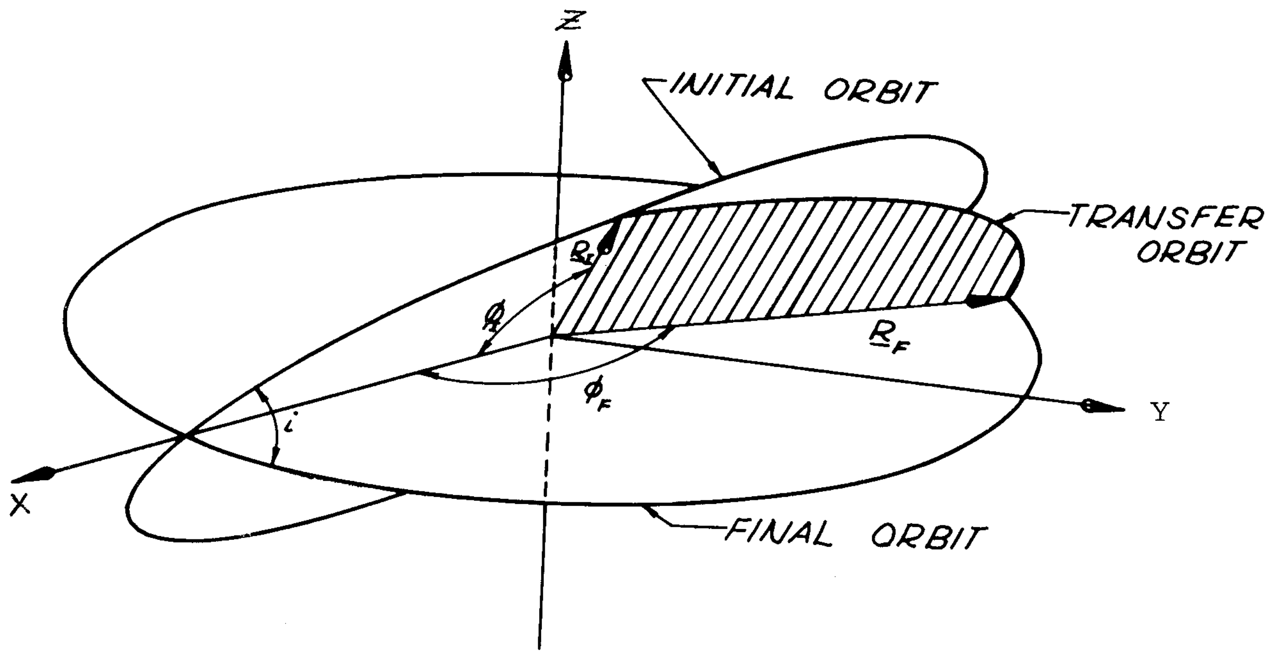


Figure 3.20 Three-Dimensional Impulsive Transfer

The work being reviewed is concerned with that class of transfers for which two impulses yield the optimum transfer; thus, the first impulse can be considered as an injection from an initial orbit into a transfer orbit, and the second as an injection from the transfer orbit to the final orbit (see Figure 3.20). The transfer orbit will, of course, intersect both the initial and final orbits, but the initial and final orbits may or may not intersect. Define an XYZ Cartesian coordinate system such that the X and Y axes are in the plane of the final orbit with the X axes along the line of intersection of the planes of the initial and final orbits. The Z axis is normal to the plane of the final orbit in the direction of the angular momentum vector. In terms of this coordinate system, the radius vector toward the departure point on the initial orbit (r_I) and the arrival point on the final orbit (r_F) are (making use of elementary equations for conic motion)

$$\underline{r_I} = \frac{p_I}{1 + e_I \cos(\phi_I - \omega_I)} \begin{bmatrix} \cos \phi_I \\ \sin \phi_I \cos i \\ \sin \phi_I \sin i \end{bmatrix} \tag{3.109}$$

$$\underline{r_F} = \frac{p_F}{1 + e_F \cos(\phi_F - \omega_F)} \begin{bmatrix} \cos \phi_F \\ \sin \phi_F \\ 0 \end{bmatrix} \tag{3.110}$$

where the subscripts I and F refer to the initial and final orbits. If the intersection of the initial orbit and the transfer orbit is used as a reference from which the angles ϕ_T and W_T are measured, then the two radius vectors can be written in terms of the transfer orbit parameters as

$$\underline{r_{IT}} = \frac{p_T}{1+e_T \cos(\phi_{IT} - \omega_T)} \begin{bmatrix} \cos(\Delta\phi - \phi_{IT}) \cos\phi_F + \sin(\Delta\phi - \phi_{IT}) \cos\alpha \sin\phi_F \\ \cos(\Delta\phi - \phi_{IT}) \sin\phi_F + \sin(\Delta\phi - \phi_{IT}) \cos\alpha \cos\phi_F \\ \sin(\Delta\phi - \phi_{IT}) \sin\alpha \end{bmatrix} \tag{3.111}$$

$$\underline{r}_{FT} = \frac{p_T}{1+e_T \cos(\phi_{FT}-\omega_T)} \begin{bmatrix} \cos(\Delta\phi-\phi_{FT})\cos\phi_F + \sin(\Delta\phi-\phi_{FT})\cos\alpha\sin\phi_F \\ \cos(\Delta\phi-\phi_{FT})\sin\phi_F + \sin(\Delta\phi-\phi_{FT})\cos\alpha\cos\phi_F \\ \sin(\Delta\phi-\phi_{FT})\sin\alpha \end{bmatrix}$$

(3.112)

where $\Delta\phi$ is the angle between $\underline{r}_I$ and $\underline{r}_F$, i.e.,

$$\Delta\phi = \arccos\left(\frac{\underline{r}_I \cdot \underline{r}_F}{|\underline{r}_I|\,|\underline{r}_F|}\right)$$

and α is the angle between the plane of the transfer orbit and the final orbit, i.e.,

$$\alpha = \arccos\left(\frac{\underline{r}_I \times \underline{r}_F \cdot \hat{z}}{\underline{r}_I \times \underline{r}_F}\right)$$

and where $\hat{z}$ is a unit vector in the z direction. Note that since the intersection of the initial and transfer orbits is taken as the reference for angles in the transfer orbit plane, the angles ϕ_{ZT} and ϕ_{FT} are

$$\phi_{IT} = 0 \quad \text{and} \quad \phi_{FT} = \Delta\phi \tag{3.113}$$

Now, equations (3.109), (3.110), (3.111) and (3.112) can be differentiated to give the corresponding velocity as

$$\overline{V}_I = \frac{p_I^2\sqrt{\frac{\mu}{p_I}}}{r_I^2(1+e_I\cos\nu_I)} \begin{bmatrix} -\sin\phi_I \\ \cos\phi_I\cos i \\ \cos\phi_I\sin i \end{bmatrix} - e_I\sin\nu_I\sqrt{\frac{\mu}{p_I}} \begin{bmatrix} \cos\phi_I \\ \sin\phi_I\cos i \\ \sin\phi_I\sin i \end{bmatrix}$$

(3.114)

$$\bar{V}_F = \frac{p_F^2\sqrt{\frac{\mu}{p_F}}}{r_F^2(1+e_F\cos\nu_F)}\begin{bmatrix}\sin\phi_F\\ \cos\phi_F\\ 0\end{bmatrix} - e_F\sin\nu_F\ \frac{\mu}{p_F}\begin{bmatrix}\cos\phi_F\\ \sin\phi_F\\ 0\end{bmatrix}$$

(3.115)

$$\bar{V}_{IT} = \frac{p_T^2\sqrt{\frac{\mu}{p_T}}}{r_I^2(1+e_T\cos\nu_{IT})}\begin{bmatrix}\sin(\Delta\phi)\cos\phi_F - \cos(\Delta\phi)\cos\alpha\sin\phi_F\\ \sin(\Delta\phi)\sin\phi_I + \cos(\Delta\phi\ \cos\alpha\cos\phi_F\\ -\cos(\Delta\phi)\sin\alpha\end{bmatrix}$$

(3.116)

$$- e\sin\nu_{IT}\sqrt{\frac{\mu}{p_T}}\begin{bmatrix}\cos(\Delta\phi)\cos\phi_F + \sin(\Delta\phi)\cos\alpha\sin\phi_F\\ \cos(\Delta\phi)\sin\phi_F + \sin(\Delta\phi)\cos\alpha\cos\phi_F\\ \sin\Delta\phi\sin\alpha\end{bmatrix}$$

(3.117)

$$\bar{V}_{FT} = \frac{p_T^2\sqrt{\frac{\mu}{p_T}}}{r_F^2(1+e_T\cos\nu_{FT})}\begin{bmatrix}-\cos\alpha\sin\phi_F\\ \cos\alpha\cos\phi_F\\ -\sin\alpha\end{bmatrix} - e_T\sin\nu_{FT}\sqrt{\frac{\mu}{p_T}}\begin{bmatrix}\cos\phi_F\\ \sin\phi_F\\ 0\end{bmatrix}$$

(3.118)

But, from equation (3.113)

$$\nu_{IT} = -\omega_T\ ,\qquad \nu_{FT} = \Delta\phi - \omega_T \qquad (3.119)$$

Thus, the characteristic velocity can be constructed as

$$\Delta V_c = |\bar{V}_I - \bar{V}_{IT}| + |\bar{V}_F - \bar{V}_{FT}| \tag{3.120}$$

Now, the orbit parameters (p, e, and ω) are known for the initial and final orbits, and, therefore, the only unknown quantities in the expressions for $\bar{V}_I$ and $\bar{V}_T$ are the true anomalies ν_I and ν_T (or equivalently, ϕ_I and ϕ_T). These quantities locate the positions at which the transfer orbit intersects the initial and final orbits. The unknown quantities of the transfer orbit are p_T, e_T, ω_T, and ν_{IT}. However, once the angles ϕ_I and ϕ_F are determined, these transfer parameters are no longer independent. In fact, ν_{IT} is given directly by equations (3.113) and (3.119). The magnitude of the radius vector is given by (3.111) and (3.112), i.e.,

$$p = r_I(1 + e\cos\omega_T)$$

$$p = r_F\left[1 + e\cos(\Delta\phi - \omega)\right]$$

These two equations can be used to reduce the number of unknown transfer orbit parameters to one. Obviously, there is a choice as to which variable is retained and the references may be consulted for the advantages of a particular variable. However, from this discussion, it is seen that the characteristic velocity for the two-impulse transfers is a function (equation (3.120)) of three variables and that a numerical technical must now be used to find the minimum value of this function.

2.3.2.3.2.2 Numerical Optimization Application. Before discussing the application of conventional gradient techniques to the solution of the optimization problem, a difficulty and one method of overcoming this difficulty used by the authors of References 3.21 and 3.22 will be considered. The difficulty is that the conventional numerical search techniques used to find the minimum of a multidimension function find only the nearest local minimum and provide no indication of the location and relative size of other minimums. The method used in References 3.21 and 3.22 to locate the minimums of the characteristic velocity function is simply to evaluate this function for a large range of the variables. These data are then used to plot curves of constant ΔV_c in the function space, and a visual determination of the location and relative size of the minima can be made. Since the function space is three-dimensional, a visualization of the space must be made by a series of two-dimensional plots which represent the trace of the ΔV_C surface on a cutting plane. An illustration of such a plot, taken from Reference 3.21 is shown in Figure 3.21. In this case, the characteristic velocity was written in terms of three variables ϕ_I, ϕ_F, and W^T. With this choice of variables, the entire function space is contained in a cube whose sides have length 2π.

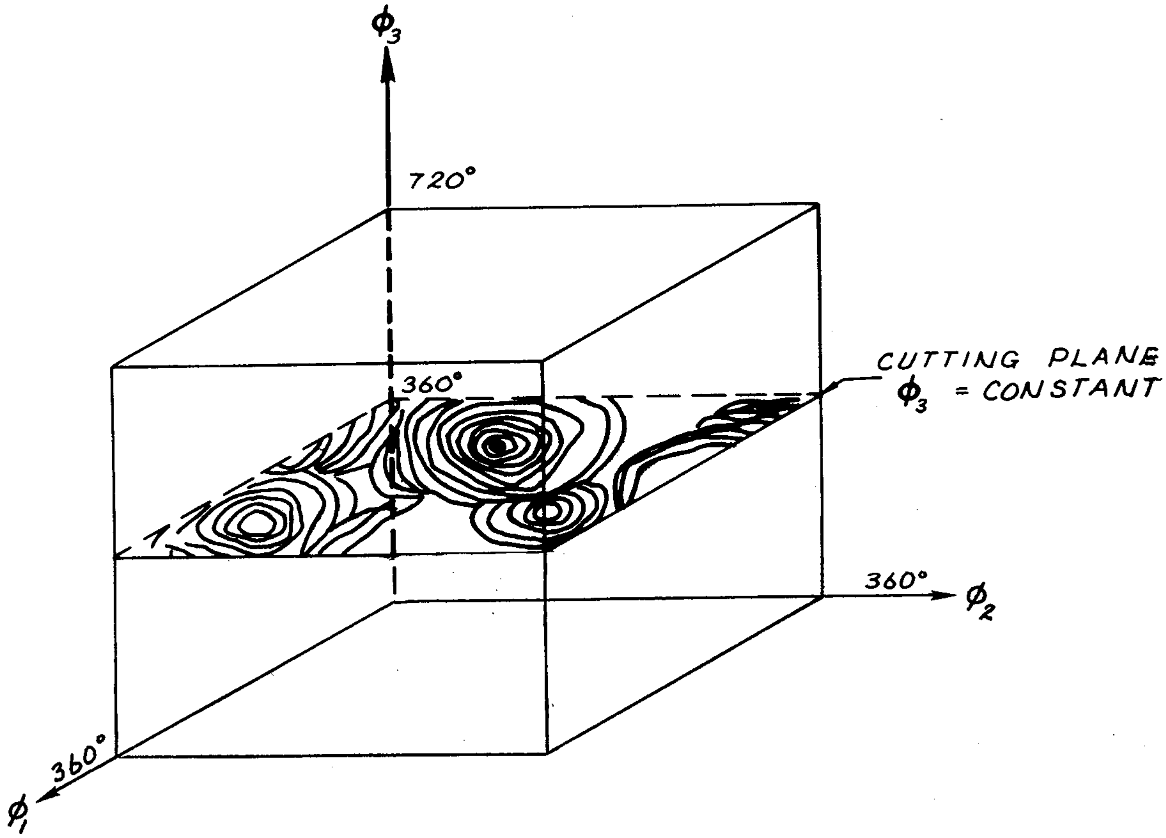


Figure 3.21 Characteristic Velocity Contours

An alternate method, and in fact, a more desirable method for very complex ΔV_c surfaces, is to choose two of the three variables at random and then minimize ΔV_c as a function of the single remaining variable. The entire range of the two variables chosen can be covered and a plot connecting equal values of ΔV_c for the optimum choice of the third parameter made. An illustration of such a plot taken from Reference 3.22 is presented in Figure 3.22. In this reference, the three variables are ϕ_I, ϕ_F, and ρ_T. The variables ϕ_I and ϕ_F are chosen at random, and then a one-dimension optimization on ρ_T is performed. The one-dimensional optimization on the variable ρ_T, i.e., the solution of the equation

$$\frac{d\,\Delta V_C\,(p)}{dp} = 0$$

leads to an eighth-order polynomial whose real roots, as shown by Lee in Reference 3.24, must include all the values of ρ_T for which the characteristic velocity is an extremum.

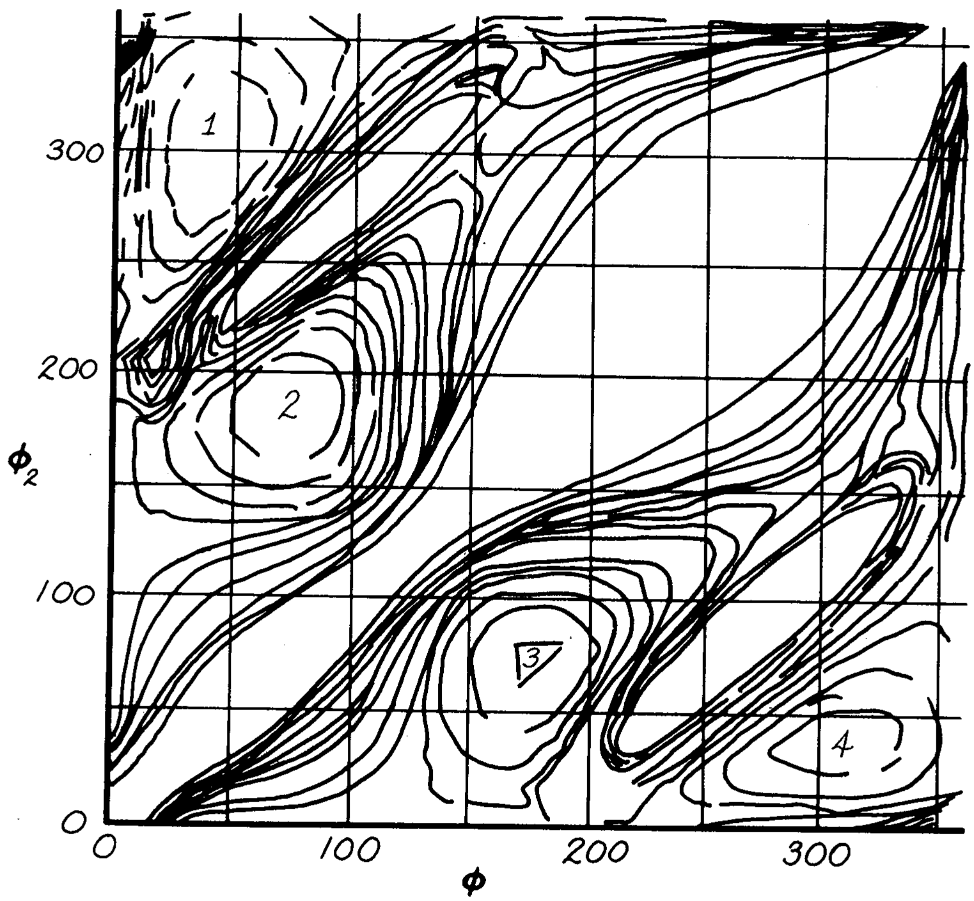


Figure 3.22 "P-Optimization" Contours

Once the approximate location of the minima have been determined by examination of the contour map, a good starting point for a numerical search technique is available, and the numerical methods can be initiated. One method for accomplishing the numerical determination of the minimum is the method of steepest descent. The basic idea of this method is that the desired solution can be found by starting at an arbitrary point in the neighborhood of the solution and stepping in the direction of the greatest change of the function, that is, along the gradient vector. Thus, if the characteristic velocity is considered a function of the three variables ϕ_1, ϕ_2, and ρ_T and an arbitrary starting point is chosen, then the changes to be made in these variables to reach a minimum (for the first step of the process) is

$$\begin{bmatrix} \Delta \phi_1 \\ \Delta \phi_2 \\ \Delta p_T \end{bmatrix} = - k \begin{bmatrix} \frac{\partial \Delta V_c}{\partial \phi_1} \\ \frac{\partial \Delta V_c}{\partial \phi_2} \\ \frac{\partial \Delta V_c}{\partial p} \end{bmatrix}$$

where k defines the size of the step taken in the (negative) gradient direction. The original starting point is updated by the left-hand side of this equation, and the updated value is used as a new starting point for another step. There are two options which may be exercised in computing the gradient for this second iteration. The first is to recompute the value of the gradient at the new starting point and proceed in this manner until the desired minimum is reached; this method is called a continuous descent method (see Reference 3.25). The second method is to proceed in the direction of the original gradient until a minimum of the function (in that direction) is located. At this point, a new gradient is computed and the process repeated. This method is called a stepwise method and obviously decreases the number of gradient calculations which must be made. Figure 3.23 from Reference 3.26 compares the continuous and stepwise revision for a two-dimensional problem.

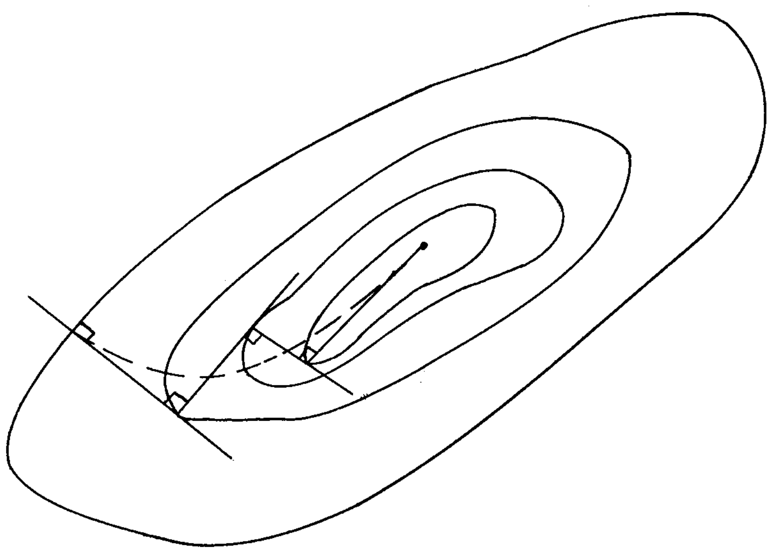


Figure 3.23 Comparison of Continuous and Stepwise Processes

As may be expected, the choise of the step size is an important consideration in using either of these methods. McCue and Hoy (Reference 3.25) adopt a procedure which varies the step size and direction in a continuous descent process at each setup and have described this approach as an "adaptive descent" process. The equation for the $n+1$ iteration of this process is

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ P_T \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ P_T \end{bmatrix} - \alpha \begin{bmatrix} \frac{S_1}{S_j} & 0 & 0 \\ 0 & \frac{S_2}{S_j} & 0 \\ 0 & 0 & \frac{S_3}{S_j} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

where α is a constant which modifies the step size, S_i are variable scaling parameters, and g_i are components of a unit vector in the gradient direction. The scaling factors are normalized relative to one of their members to provide a reference magnitude for determination of the step size. The logic for selecting the step size on scaling parameters is as follows:

(1) If the inequality

$$\Delta V_{c_{n+1}} < \Delta V_{c_n} \tag{3.121}$$

is not satisfied, α is decreased by a prescribed percentate, and a new coordinate vector $(\phi_1, \phi_2, P_T)_{n+1}$ is computed. This n^{th} stage process is repeated until the inequality is satisfied or $\alpha \leq E_2$. When $\alpha \leq E$ sufficient convergence has been achieved and the process is terminated.

(2) If equation (3.121) is satisfied, the $(n+1)$ coordinate vector is adopted and the n^{th} step is considered complete. If equation (3.121) is satisfied during each of a number of steps, this successful behavior is rewarded by increasing α by a prescribed percentage.

(3) Since the scaling parameters are normalized, only their relative size is of interest. To produce a change in relative size, undesirable behavior can be penalized by decreasing a given scale factor by a predetermined percentage. In practice, a scaling parameter (S_j) is decreases each time the corresponding component of the gradient vector (g_j) changes sign. No provision for increasing S_j is required since decreasing S_i $(i\ j)$ automatically causes S_j to become relatively more influential. McCue and Hoy applied this adaptive descent method to a pair of orbits described in Table 3.2 with the resulting transfer orbits given in Table 3.3. The four minima indicated in Table 3.3 are those indicated in Figure 3.22 which also refers to the orbits of Table 3.2.

	p (mi)	e	(deg)	i(deg)
INITIAL ORBIT	5000	0.2	-90.0	5.0
FINAL ORBIT	6000	0.2	30.0	0

TABLE 3.2 TERMINAL ORBIT CHARACTERISTICS

1	73.8152	187.5568	6644.8496	4902.651223852	3.4
2	40.8343	298.2634	6617.7904	5343.148693477	2.5
3	177.8114	73.6465	4611.8023	5393.781144757	3.0
4	308.2034	37.7403	4592.8574	5654.191209679	2.8

TABLE 3.3 TRANSFER ORBIT CHARACTERISTICS

A comparison between the adaptive descent and the stepwise descent was made for the long narrow "valley" region of the characteristic velocity contour of Figure 3.22 with the result given in Figure 3.23. This figure indicates the superiority of the adaptive descent method in the area of a difficult to locate minima. Further numerical results can be found in References 3.27, 3.28 and 3.29.

3.0 RECOMMENDED PROCEDURES

The previous sections of this monograph have illustrated the use of the three optimization techniques developed in other monographs of this series by applying them to problems encountered in space flight. Although both the Calculus of Variations and the Pontryagin Maximum Principle have been applied to the various problems, it was pointed out in Section 2.1 that these two formulations yield the same results in terms of the equations which must be solved and, therefore, could be considered to be different means of looking at one formulation of an optimization technique. While the Calculus of Variations or the Pontryagin Maximum Principle lead to a set of first-order ordinary differential equations (for the Mayer-type problem), the application of Dynamic Programming leads to a set of partial differential equations. Thus a classification of optimization techniques which groups the Calculus of Variations and the Pontryagin Maximum Principle together in one group and Dynamic Programming in the other group could be made. (The illustration of the use of Dynamic Programming given in Section 2.3.1.4 indicated that the same result could be obtained by the Pontryagin Maximum Principle. However, it is not surprising that the results obtained by using these two techniques are the same since it could be shown that both formulations admitted to an analytic solution. The fact that the control in both cases had the same form does not indicate the same relation between Dynamic Programming and the Pontryagin Maximum Principle as exists between the Calculus of Variations and the Pontryagin Maximum Principle.) It is generally felt that on a theoretical level Dynamic Programming is not as strong or as generally applicable as the Calculus of Variations or the Pontryagin Maximum Principle, however, since it is a completely different approach to the problem, its use could provide different perspective and insight to the structure of the problem under consideration.

It is difficult to recommend a general procedure for formulating optimization problems since each of the three methods illustrated in this monograph have their own critics and extollers, and since the choice of a particular method is dictated by the analyst's personal familiarity with the three techniques. As mentioned earlier, it is generally felt that Dynamic Programming is not as strong theoretically as the other two methods, nor is there as much experience with Dynamic Programming for trajectory type problems as there is with the Calculus of Variations or the Pontryagin Maximum Principle. For these reasons the Dynamic Programming approach might be considered as an experimental technique whose use may provide new insight into the problem. As for the choice between the Calculus of Variations and the Pontryagin Maximum Principle, there is little to distinguish the two on the basis of information attainable about the problem. However, the Pontryagin Maximum Principle seems to be more suited to the types of problems (i.e., the way the problems are formulated) encountered in guidance and trajectory work, and for that reason, it can generally be recommended over the Calculus of Variations.

As was illustrated in several examples of the previous sections, the method of attacking an optimization problem using the Pontryagin Maximum Principle formulation is summarized below.

1. Write the differential constraint equations as a set of first order differential equations (by defining new state variables if necessary).

$$\underline{X} = \underline{f}(\underline{X}, \underline{U})$$

where the vector X is the state vector and the vector U is the control vector

2. Form the hamiltonian

$$H \triangleq P^T f(\underline{X}, \underline{U})$$

and co-state equations

$$\dot{\underline{P}} = -\frac{\partial H}{\partial \underline{X}}$$

3. Maximize the hamiltonian with respect to the components of the control vector - thereby determining the optimal control U^* in terms of $\underline{X}$ and $\underline{P}$

$$H(X, P, U^*) \geq H(X, P, U)$$

4. Substitute the optimal control in the state equations

Generally, the co-state variables cannot be eliminated from the control equations by the use of an analytic solution so that numerical methods must be employed. A further complicating factor is that in many problems of interest the known boundary conditions are divided between the initial and terminal time so that a two point boundary value problem must be solved.

Thus, another choice must now be made as to which numerical method must be used. Again there is not clearcut superiority of one method over another with the choice based on the characteristic of the particular problem and the desirability of rapid convergence (Newton-Raphson or other second order method) vs the radius of convergnece (gradient methods). In general, the difficulties associated with these iterative techniques (e.g. slow convergence or lack of convergence) rule out their use as part of a real time feedback mechanization; thus, if optimization of such a closed loop system is desired, the use of a "nominal" optimum solution as a basis for a perturbative scheme with a closed form solution, and/or the use of a cost function which only approximates the actual cost (e.g. the quadratic cost function as discussed in Section 2.3.2.2.3 or the penalty function approach of Section 2.3.1.4.1) but allows the determination of an analytic expression for the control should be investigated.

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